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Chern classes in Deligne cohomology for coherent analytic sheaves

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Abstract In this article, we construct Chern classes in rational Deligne cohomology for coherent sheaves on a smooth complex compact manifold. We prove that these classes satisfy the functoriality property under pullbacks, the Whitney formula and the Grothendieck–Riemann–Roch theorem for projective morphisms between smooth complex compact manifolds.

1 Introduction

Let X be a smooth differentiable manifold and E be a complex vector bundle of rank r on X. By the Leray–Hirsch theorem, if $\xi = c_1(\mathcal{O}_E(1)) \in H^2(\mathbb{P}(E), \mathbb{Z})$, then $H^*(\mathbb{P}(E), \mathbb{Z})$ is a free module over $H^*(X, \mathbb{Z})$ with basis $1, \xi, \dots, \xi^{r-1}$. The topological Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$ are defined by the relation

$$\xi^r + \pi^* c_1(E)\xi^{r-1} + \cdots + \pi^* c_r(E) = 0,$$

where $\pi : \mathbb{P}(E) \to X$ is the canonical projection.

This method produces Chern classes in many contexts, provided that the first Chern class is already defined and that there is a structure theorem for the cohomology of a projective bundle (see [16]). Under mild assumptions on the cohomology ring (Axioms A in Sect. 2.1), the total Chern class $c(E) = 1 + c_1(E) + \cdots + c_r(E)$ is functorial by pullback and multiplicative under exact sequences.

If X is a smooth projective variety over $\mathbb{C}$ and E is an algebraic vector bundle on X, the classes $c_i(E)$ in $CH^i(X)$ are obtained by this construction. If $\mathcal{F}$ is an algebraic

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coherent sheaf on X, there exists a resolution

$$0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_N \longrightarrow \mathcal{F} \longrightarrow 0$$

of $\mathcal{F}$ by locally free sheaves. The total Chern class $c(\mathcal{F})$ is defined by

$$c(\mathcal{F}) = c(E_N)\, c(E_{N-1})^{-1} c(E_{N-2}) \dots$$

and is independent from the chosen resolution [5]. Therefore, the theory of Chern classes for coherent sheaves on smooth projective manifolds is a purely formal consequence of the theory for locally free sheaves.

Although locally free resolutions exist for coherent sheaves on curves and complex surfaces [24], this is no longer true for higher-dimensional complex manifolds, as shown by Voisin.

Theorem [28] *On any generic complex torus of dimension greater than* 3*, the ideal sheaf of a point does not admit a global locally free resolution.*

The classical approach fails due to the lack of global resolutions. Nevertheless, some constructions have been carried out in specific cohomology rings:

- The integer cohomology ring $H^*(X, \mathbb{Z})$. This is done by the Grauert vanishing theorem [13] using real analytic locally free resolutions instead of holomorphic ones.
- The Dolbeault cohomology ring $\oplus_i H^i(X, \Omega^i_X)$. This has been done in [1] using extension classes.
- Atiyah's method has been generalized in [14] using results of [6]. It produces Chern classes in the analytic de Rham cohomology ring $\oplus_i \mathbb{H}^i(X, \Omega_X^{\bullet \geq i})$ (see [26]).

In this article, our aim is to construct Chern classes for coherent analytic sheaves on a smooth complex compact manifold with values in cohomology theories satisfying specific axioms. These cohomology theories are $\mathbb{Q}$-vector spaces, so that we do not take account of torsion phenomena.

Our main result is the following:

Theorem 1 *Let* $X \mapsto A(X)$ *be a cohomology theory for smooth complex manifolds which satisfies Axioms B of Sect.* 2.1*. If* X *is compact and* $G(X)$ *is the Grotendieck ring of coherent sheaves on* X*, we can define a Chern character* $\mathrm{ch}\colon G(X) \to A(X)$ *such that*

(i) ch *is functorial by pullback under holomorphic maps.*
(ii) ch *is an extension of the usual Chern character for locally free sheaves.*
(iii) *The Grothendieck–Riemann–Roch theorem holds for projective morphisms between smooth complex compact manifolds.*

Our method produces a complete characterization of a theory of Chern classes:

Theorem 2 *Let* $X \mapsto A(X)$ *be a cohomology theory on smooth complex compact manifolds which satisfies Axioms C in Sect.* 2.1*. Let* $\mathrm{ch}, \mathrm{ch}'\colon G(X) \to A(X)$ *be two group morphisms such that*

(i) ch *and* ch′ *are functorial by pullback under holomorphic maps.*
(ii) *For every line bundle* L, $\mathrm{ch}(L) = \mathrm{ch}'(L)$.
(iii) ch *and* ch′ *verify the Grothendieck–Riemann–Roch theorem for smooth immersions.*

Then $\mathrm{ch} = \mathrm{ch}'$.

As an application of these results, we obtain a Chern character for coherent sheaves with values in the rational Deligne cohomology ring $A(X) = \oplus_i H_D^{2i}(X, \mathbb{Q}(i))$ satisfying the Grothendieck–Riemann–Roch theorem for projective morphisms. These classes are compatible with the topological and Atiyah Chern classes. The compatibility with the Green Chern classes remains unknown in the non Kähler case (see Sect. 5.2).

Our construction of the Chern character (Theorem 1) is achieved by induction on $\dim X$ in Sect. 3. The case of torsion sheaves is settled by the Grothendieck–Riemann–Roch theorem if $\mathcal{F}$ is supported in a smooth hypersurface. Then, we reduce the general case to the former one by dévissage and blowups. For sheaves of positive generic rank, we can suppose after taking a bimeromorphic model of X that $\mathcal{F}/\mathcal{F}_{\mathrm{tor}}$ is locally free. This is the key property to define $\mathrm{ch}(\mathcal{F})$ for arbitrary coherent sheaves. In Sect. 4, we prove that the Chern character constructed in Sect. 3 is additive under short exact sequences. This is done by deformation of the extension class after several simplifications obtained by blowups. In Sect. 5, we prove the Grothendieck–Riemann–Roch theorem for arbitrary projective morphisms. Then we prove Theorem 2 and discuss compatibility results. The axiomatic setup for cohomology rings is explained in Sect. 2 and will be used throughout the article.

2 Cohomology theories and Chern classes for locally free sheaves

2.1 Axiomatic cohomology theory

In this section, we introduce three sets of axioms for arbitrary cohomology rings on smooth complex manifolds. We assume to be given for each smooth complex manifold X a graded commutative cohomology ring $A(X) = \oplus_{i=0}^{\dim X} A^i(X)$ which is an algebra over $\mathbb{Q} \subset A^0(X)$.

Axioms A

(i) For each holomorphic map $f: X \to Y$, there exists a functorial pullback morphism $f^*: A^*(Y) \to A^*(X)$, compatible with the products and the gradings.
(ii) A functorial group morphism $c_1: \mathrm{Pic}(X) \to A^1(X)$ is given.
(iii) If E is a holomorphic vector bundle of rank r on X, then $A(\mathbb{P}(E))$ is a free module over $A(X)$ with basis $1, c_1(\mathcal{O}_E(1)), \ldots, c_1(\mathcal{O}_E(1))^{r-1}$.
(iv) If X is covered by two open sets U and V, then the product map
$$\ker(\, A(X) \to A(U)\,) \otimes_{\mathbb{Q}} \ker(\, A(X) \to A(V)\,) \longrightarrow A(X) \text{ identically}$$
vanishes.

Property (iii) implies the $\mathbb{P}^1$-homotopy principle:

$$\forall t \in \mathbb{P}^1(\mathbb{C}), \quad j_t^*: A^*(X \times \mathbb{P}^1) \to A^*(X \times \{t\}) \simeq A^*(X)$$

is independent from t.

If $X \mapsto A(X)$ satisfies Axioms A, then for every holomorphic vector bundle E on X it is possible to define classes $c_i(E) \in A^i(X)$ by the relation $\xi^r + \pi^* c_1(E)\, \xi^{r-1} + \cdots + \pi^* c_r(E) = 0$, where $\pi : \mathbb{P}(E) \to X$ is the projection and $\xi = c_1(\mathcal{O}_E(1)) \in A^1(\mathbb{P}(E))$. The classes $c_i(E)$ are clearly functorial by pullbacks under holomorphic maps. The total Chern class of E is $c(E) = 1 + c_1(E) + \cdots + c_r(E)$. The property (iv) assures that $c(E \oplus F) = c(E)c(F)$, which is the Whitney formula in the split case (see [27, Ch.11, Sect. 2]). The general case can be reduced to the split case by deforming the extension class of the exact sequence.

Let us briefly recall the construction of the exponential Chern classes. The symmetric sums and the Newton sums in the formal variables $x_1, \ldots, x_r$ are defined by:

$$\prod_{i=1}^{r}(X + x_i) = \sum_{i=0}^{r} \sigma_i(x_1, \ldots, x_r) X^{r-i}, \qquad \sigma_k(x_1, \ldots, x_r) = 0 \text{ if } k > r;$$
$$S_k(x_1, \ldots, x_r) = \frac{1}{k!}(x_1^k + \cdots + x_r^k).$$

For all $n > 0$, there exist $P_n,\ Q_n \in \mathbb{Q}[T_1, \ldots, T_n]$ characterized by the following identities: $\forall\, r,\ n \geq 1$, if $\underline{x} = (x_1, \ldots, x_r)$, then $S_n(\underline{x}) = P_n(\sigma_1(\underline{x}), \ldots, \sigma_n(\underline{x}))$ and $\sigma_n(\underline{x}) = Q_n(S_1(\underline{x}), \ldots, S_n(\underline{x}))$. The *exponential Chern classes* $\mathrm{ch}_i(E)$ are defined by $\mathrm{ch}_0(E) = \mathrm{rank}(E)$ and $\mathrm{ch}_i(E) = P_i(c_1(E), \ldots, c_n(E))$ for $1 \leq i \leq \dim X$. The *total exponential Chern class*, also called *Chern character*, is $\mathrm{ch}(E) = \mathrm{ch}_0(E) + \cdots + \mathrm{ch}_n(E)$. The morphism ch is additive under exact sequences. Furthermore, it satisfies the property $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$. The usual Chern classes can be recovered from the exponential ones by the relations $c_0(E) = 1$ and $c_i(E) = Q_i(\mathrm{ch}_1(E), \ldots, \mathrm{ch}_n(E))$. The situation will remain the same for coherent sheaves, except that $\mathrm{ch}_0(\mathcal{F})$ will be the generic rank of $\mathcal{F}$.

In order to avoid confusions we will use from now on the notation $\overline{\mathrm{ch}}$ instead of ch for locally free sheaves.

Axioms B

- Axioms A are satisfied.
- If $f : X \to Y$ is a proper holomorphic map of relative dimension d, there is a functorial Gysin morphism $f_*: A^*(X) \to A^{*-d}(Y)$ satisfying the following properties:

(i) The projection formula holds: $\forall x \in A(X), \forall y \in A(Y)$, $f_*\left(x \,.\, f^* y\right) = f_* x \,.\, y$.

(ii) Consider the following cartesian diagram, where p and q are the projections on the first factors:

$$\begin{array}{ccc} Y\times Z & \xhookrightarrow{i_{Y\times Z}} & X\times Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xhookrightarrow[i_Y]{} & X \end{array}$$

Then $q^*i_{Y*} = i_{Y\times Z*}\, p^*$.

(iii) Consider the cartesian diagram, where Y and Z are compact and intersect transversally in X:

$$\begin{array}{ccc} W & \xhookrightarrow{i_{W/Y}} & Y \\ {\scriptstyle i_{W/Z}}\downarrow & & \downarrow{\scriptstyle i_Y} \\ Z & \xhookrightarrow[i_Z]{} & X \end{array}$$

Then $i_Y^*\, i_{Z*} = i_{W/Y*} i^*_{W/Z}$.

(iv) Let $f: X\to Y$ be a surjective map between smooth complex compact manifolds, and let D be a smooth hypersurface of Y such that $f^{-1}(D)$ is a simple normal crossing divisor. Let us write $f^*D = m_1\widetilde{D}_1 + \cdots + m_N\widetilde{D}_N$. Let $\overline{f}_i : \widetilde{D}_i \to D$ be the restriction of f to $\widetilde{D}_i$. Then

$$f^*\, i_{D*} = \sum_{i=1}^{N} m_i\, i_{\widetilde{D}_i*}\overline{f}_i{}^*.$$

(v) Let X be compact, smooth, and let Y be a smooth proper submanifold of codimension d of X. Let $\widetilde{X}$ be the blowup of X along Y, as shown in the following diagram,

$$\begin{array}{ccc} E & \xrightarrow{j} & \widetilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[i]{} & X \end{array}$$

and let $N^*_{Y/X} = i^*\big[\mathcal{I}_Y/_{\mathcal{I}_Y^2}\big]$ and $N^*_{E/X} = j^*\big[\mathcal{I}_E/_{\mathcal{I}_E^2}\big]$ be the conormal bundles of Y and E in X and $\widetilde{X}$, respectively. Then

- the map p^* is injective,

- a class $\alpha \in A^*(\widetilde{X})$ is in the image of p^* if and only if the class $j^*\alpha$ is in the image of q^*,
- if F is the excess conormal bundle of q defined by the exact sequence

$$0 \longrightarrow F \longrightarrow q^* N^*_{Y/X} \longrightarrow N^*_{E/\widetilde{X}} \longrightarrow 0,$$

we have the excess formula: $\forall \alpha \in A(Y)$, $p^* i_* \alpha = j_* \big(q^*\alpha \,.\, c_{d-1}(F^*) \big)$.

(vi) If Y is a smooth hypersurface of X, then $\forall \alpha \in A(Y)$, $i_Y^* i_{Y*}\alpha = \alpha \,.\, c_1(N_{Y/X})$.

(vii) The Hirzebruch–Riemann–Roch theorem holds for $X = \mathbb{P}^N$, $\mathcal{F} = \mathcal{O}(i)$, $i \in \mathbb{Z}$.

Remark 1 We do not impose purity properties as in other axiomatic cohomology theories (e.g. [12]).

Axioms C

(i) For each holomorphic map $f : X \to Y$, there exists a functorial pullback morphism $f^* : A(Y) \to A(X)$, compatible with the products and the gradings.

(ii) If σ is the blowup of a smooth complex compact manifold along a smooth proper submanifold, then σ^* is injective.

(iii) If E is a holomorphic vector bundle on X and $\pi : \mathbb{P}(E) \to X$ is the projection of the associated projective bundle, then π^* is injective.

(iv) If X is a smooth complex compact manifold and Y is a smooth submanifold of codimension d, then there is a Gysin morphism $i_* : A^*(Y) \to A^{*+d}(X)$.

2.2 Deligne cohomology

If X is a smooth complex manifold, we can consider its Deligne cohomology groups $H_D^{2i}(X, \mathbb{Z}(i))$ (see [9]). This is one of the most refined cohomology theory known in the non-algebraic context (in the non-Kähler case, a more refined one has been constructed in [25]). Our aim in this section is to prove that the rational Deligne cohomology groups $A^i(X) = H_D^{2i}(X, \mathbb{Q}(i))$ satisfy Axioms B. For the classical properties of Deligne cohomology, we refer to [9]. The construction of a Gysin morphism is sketched in [10].

Proposition 1 *If $A^i(X) = H_D^{2i}(X, \mathbb{Q}(i))$, then $X \mapsto A(X)$ satisfies Axioms B.*

Proof Axioms A are clearly satisfied [for (ii), remark that $H_D^2(X, \mathbb{Z}(1)) = \mathrm{Pic}(X)$]. The formulae (iii) and (iv) are of the same type. Let us prove (iv). We first define some notations: let Γ be the graph of $i_D : D \to Y$ and Γ_i be the graph of $i_{\widetilde{D}_i} : \widetilde{D}_i \to X$. We define $\Gamma_i' = \big(\overline{f}_i, \mathrm{id}\big)_* \big(\Gamma_i\big) \subseteq D \times X$. We call $p_1 : D \times Y \to D$ and $p_2 : D \times Y \to Y$ the first and second projections. Similarly, we define the projections $p_1{}' : D \times X \to D$, $p_2' : D \times X \to X$, $p'_{1,i} : \widetilde{D}_i \times X \to \widetilde{D}_i$, and $p'_{2,i} : \widetilde{D}_i \times X \to X$. Using explicit description of the Bloch cycle class (see [4]), we have $(\mathrm{id}, f)^*\{\Gamma\} = \sum_{i=1}^N m_i\,\{\Gamma'_i\}$. Then

$$\begin{aligned} f^*\,i_{D*}\,\alpha &= f^*\,p_{2*}\left(p_1^*\alpha\,.\,\{\Gamma\}\right) = p'_{2*}(\mathrm{id}, f)^*\left(p_1^*\,\alpha\,.\,\{\Gamma\}\right)\\ &= \sum_{i=1}^N m_i\,p'_{2*}\left(p'^{\,*}_1\alpha\,.\,\{\Gamma'_i\}\right) = \sum_{i=1}^N m_i\,p'_{2*}\left(p'^{\,*}_1\alpha\,.\,(\overline{f}_i, \mathrm{id})_*\{\Gamma_i\}\right)\\ &= \sum_{i=1}^N m_i\,p'_{2,i*}\left(p'^{\,*}_{1,i}\,\overline{f}_i^{\,*}\alpha\,.\,\{\Gamma_i\}\right) = \sum_{i=1}^N m_i\,i_{\widetilde{D}_i*}\,\overline{f}_i^{\,*}\,\alpha, \end{aligned}$$

by (i), (ii) and the projection formula.

In the case of étale cohomology, it is possible to assume in (vi) that $\alpha = 1$ (see [17, Exposé VII, Sect. 4] and [7, Cycle Sect. 1.2]). Remark that this is no longer possible here, for there is no purity theorem in Deligne cohomology.

We use the deformation to the normal cone (see [11, Ch. 5, Sect. 5.1], [21] and [17, Exposé VII, Sect. 9]). Let $M_{Y/X}$ be the blowup of $X \times \mathbb{P}^1$ along $Y \times \{0\}$, $\widetilde{X} = X$ be the blowup of X along Y, and $M^\circ_{Y/X} = M_{Y/X}\backslash\widetilde{X}$. Then we have an injection $F : Y \times \mathbb{P}^1 \rightarrow M^\circ_{Y/X}$ over $\mathbb{P}^1$. We denote the inclusions $N_{Y/X} \rightarrow M^\circ_{Y/X}$ and $Y \rightarrow N_{Y/X}$ by j_0 and i, the projections of $\left(Y \times \mathbb{P}^1\right) \times M^\circ_{Y/X}$ (resp. $Y \times \mathbb{P}^1$, resp. $\left(Y \times \mathbb{P}^1\right) \times N_{Y/X}$, resp. $Y \times N_{Y/X}$) on its first and second factor by pr_1 and pr_2 (resp $\widetilde{\mathrm{pr}}_1$ and $\widetilde{\mathrm{pr}}_2$, resp. pr'_1 and pr'_2, resp. pr''_1 and pr''_2). Besides, $\Gamma \subseteq Y \times \mathbb{P}^1 \times M^\circ_{Y/X}$ is the graph of F and $\Gamma' \subseteq Y \times N_{Y/X}$ is the graph of i. Finally, $\Gamma'' = \left(i_0, \mathrm{id}_{N_{Y/X}}\right)_*\Gamma' \subseteq Y \times \mathbb{P}^1 \times N_{Y/X}$, where $i_0 : Y \times \{0\} \rightarrow Y \times \mathbb{P}^1$ is the injection of the central fiber. Remark that pr'_2 and pr''_2 are proper maps since Y is compact.

We have $\left(i_0, \mathrm{id}_{N_{Y/X}}\right)_*\{\Gamma'\} = \{\Gamma''\}$ and $\left(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0\right)^*\{\Gamma\} = \{\Gamma''\}$. Let γ be the class on $M^\circ_{Y/X}$ defined by $\gamma = F_*\left(\widetilde{\mathrm{pr}}_1^*\alpha\right)$. Then, by (ii) and the projection formula,

$$\begin{aligned} j_0^*\gamma &= j_0^*\,\mathrm{pr}_{2*}\left(\mathrm{pr}_1^*\widetilde{\mathrm{pr}}_1^*\alpha\,.\,\{\Gamma\}\right) = \mathrm{pr}'_{2*}\left[\left(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0\right)^*\left(\mathrm{pr}_1^*\widetilde{\mathrm{pr}}_1^*\alpha\,.\,\{\Gamma\}\right)\right]\\ &= \mathrm{pr}'_{2*}\left[\left(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0\right)^*\ \mathrm{pr}_1^*\widetilde{\mathrm{pr}}_1^*\alpha\,.\,\{\Gamma''\}\right]\\ &= \mathrm{pr}'_{2*}\left(i_0, \mathrm{id}_{N_{Y/X}}\right)_*\left[\left(i_0, \mathrm{id}_{N_{Y/X}}\right)^*\ \left(\mathrm{id}_{Y\times\mathbb{P}^1}, j_0\right)^*\ \mathrm{pr}_1^*\widetilde{\mathrm{pr}}_1^*\alpha\,.\,\{\Gamma'\}\right]\\ &= \mathrm{pr}''_{2*}\left(\mathrm{pr}''^*_1\alpha\,.\,\{\Gamma'\}\right) = i_*\alpha. \end{aligned}$$

By the homotopy principle, the class $F^*\gamma_{|Y\times\{t\}}$ is independent from t. If $t \neq 0$, we have clearly $F^*\gamma_{|Y\times\{t\}} = i_Y^*i_{Y*}\alpha$. For $t = 0$, $F^*\gamma_{|Y\times\{0\}} = i^*j_0^*\gamma = i^*i_*\alpha$. Let $\pi : N_{Y/X} \rightarrow Y$ be the projection of $N_{Y/X}$ on Y. Then $\alpha = i^*\pi^*\alpha$ and therefore $i^*i_*\alpha = i^*i_*(i^*\pi^*\alpha) = i^*\left(\pi^*\alpha\,.\{\overline{Y}\}\right) = \alpha\,.\,i^*\{\overline{Y}\}$, where $\{\overline{Y}\}$ is the cycle class of Y in $N_{Y/X}$. Now $i^*\{\overline{Y}\} = c_1\left(N_{Y/N_{Y/X}}\right) = c_1\left(N_{Y/X}\right)$, and the proof is complete.

We can now prove (v). By dévissage, we have an isomorphism

$$
\begin{aligned}
H_D^*(X) \oplus \bigoplus_{i=1}^{d-1} H_D^*(Y) &\longrightarrow H_D^*(\widetilde{X}) \\
\Big(x, (y_i)_{1\le i\le d-1}\Big) &\longmapsto p^*x + \sum_{i=1}^{d-1} j_*\left[y_i\, c_1\left(\mathcal{O}_{N_{Y/X}}(-1)\right)^{i-1}\right].
\end{aligned}
$$

The injectivity of p^* is clear, since $p_*p^* = \mathrm{id}$. If α is a Deligne class on $\widetilde{X}$, we can write $\alpha = p^*x + \sum_{i=1}^{d-1} j_*\big[y_i\, c_1\big(\mathcal{O}_{N_{Y/X}}(-1)\big)^{i-1}\big]$. Since E is a hypersurface of $\widetilde{X}$, by the formula proved above $j^*j_*\lambda = \lambda\, c_1(N_{E/\widetilde{X}}) = \lambda\ c_1(\mathcal{O}_{N_{Y/X}}(-1))$ for any Deligne class λ on E. We obtain $j^*\alpha = q^*i^*x + \sum_{i=1}^{d-1}(-1)^i y_i\, c_1\big(\mathcal{O}_{N_{Y/X}}(1)\big)^i$. If $j^*\alpha = q^*\delta$, all the classes y_i vanish. Thus $\alpha = p^*x$. For the excess formula, let α be a Deligne class on Y. We define $\beta = j_*\left(q^*\alpha\ c_{d-1}(F^*)\right)$. Then, by (vi), $j^*\beta = \left[q^*\alpha\ c_{d-1}(F^*)\right]\, c_1\big(N_{E/\widetilde{X}}\big) = q^*\big[\alpha\, c_d\big(N_{Y/X}\big)\big]$. Therefore, β comes from the base so that $\beta = p^*p_*\beta = p^*i_*q_*\left(q^*\alpha\, c_{d-1}(F^*)\right) = p^*i_*\left[\alpha\ q_*\left(c_{d-1}(F^*)\right)\right] = p^*i_*\alpha$ for $q_*\left(c_{d-1}(F^*)\right) = 1$ (see [5, Lemme 19.b]). □

3 Construction of Chern classes

From now on, we will consider a cohomology theory $X \mapsto A(X)$ which satisfies Axioms B. If X is compact, we define $G(X)$ as the Grothendieck ring of coherent sheaves on X. The class of a coherent sheaf $\mathcal{F}$ in $G(X)$ will be denoted by $[\mathcal{F}]$. The construction of the total exponential Chern class $\mathrm{ch}(\mathcal{F})$ in $A(X)$ for an arbitrary coherent sheaf $\mathcal{F}$ on X will be done by induction on $\dim X$.

Let us precisely state the induction hypothesis (H_n):
For any complex compact manifold X of dimension at most n and any coherent analytic sheaf $\mathcal{F}$ on X, we assume to be given a class $\mathrm{ch}(\mathcal{F})$ in $A(X)$ satisfying the following properties:

- (W_n) If $\dim X \le n$ and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent analytic sheaves on X, then $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$.
- (F_n) If $\dim X,\ \dim Y \le n$ and $f\colon X \to Y$ is a holomorphic map, then $\forall y \in G(Y)$, $\mathrm{ch}(f^!y) = f^*\,\mathrm{ch}(y)$.
- (C_n) If $\dim X \le n$ and $\mathcal{F}$ is locally free, $\mathrm{ch}(\mathcal{F}) = \overline{\mathrm{ch}}(\mathcal{F})$.
- (P_n) If $\dim X \le n$, $\mathrm{ch}(1) = 1$ and $\forall x,\ y \in G(X)$, $\mathrm{ch}(x\,.\,y) = \mathrm{ch}(x)\,\mathrm{ch}(y)$.
- (RR_n) If Z is a smooth hypersurface of X, where $\dim X \le n$, then for every coherent sheaf $\mathcal{F}$ on Z, $\mathrm{ch}(i_{Z*}\mathcal{F}) = i_{Z*}\big(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1}\big)$.

From now on, we will suppose that the induction hypothesis (H_{n-1}) above is true.

Theorem 3 *Assuming hypothesis* (H_{n-1})*, we can define a Chern character for coherent sheaves on compact complex manifolds of dimension n, which satisfies* (H_n).

Let us briefly explain the organization of the proof of this theorem. In Sect. 3.1, we construct the Chern character for coherent torsion sheaves. In Sect. 3.3, we construct the Chern character for arbitrary coherent sheaves, using the result of Sect. 3.2. Properties (RR_n) and (C_n) will be obvious consequences of the construction. In Sect. 4.3, we prove (W_n) and then (F_n) and (P_n) using the preliminary results of Sect. 4.1 and Sect. 4.2.

3.1 Construction for torsion sheaves

In this section, we define a Chern character for torsion sheaves by forcing the Grothendieck–Riemann–Roch theorem for immersions of smooth hypersurfaces. Let $G_{\mathrm{tors}}(X)$ denote the Grothendieck group of the abelian category of coherent torsion sheaves on X. We will prove the following version of Theorem 3 for torsion sheaves:

Proposition 2 *On any n-dimensional complex manifold we can define a Chern character for torsion sheaves such that*:

(i) (W_n) *If* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *is an exact sequence of torsion sheaves on X and if* $\dim X \leq n$, *then* $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$.
(ii) (P_n) *Let $\mathcal{E}$ be a locally free sheaf and $x \in G_{\mathrm{tors}}(X)$. Then* $\mathrm{ch}([\mathcal{E}].x) = \overline{\mathrm{ch}}(\mathcal{E}) \,.\, \mathrm{ch}(x)$.
(iii) (F_n) *Let $f : X \to Y$ be a holomorphic map where* $\dim X \leq n$ *and* $\dim Y \leq n$, *and $\mathcal{F}$ be a coherent sheaf on Y such that $\mathcal{F}$ and $f^*\mathcal{F}$ are torsion sheaves. Then* $\mathrm{ch}(f^![\mathcal{F}]) = f^*\,\mathrm{ch}(\mathcal{F})$.
(iv) (RR_n) *If Z is a smooth hypersurface of X and $\mathcal{F}$ is coherent on Z,*

$$\mathrm{ch}\left(i_{Z*}\mathcal{F}\right) = i_{Z*}\left(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1}\right).$$

We will proceed in three steps. In Sect. 3.1.1, we perform the construction for coherent sheaves supported in a smooth hypersurface. In Sect. 3.1.2, we deal with sheaves supported in a simple normal crossing divisor. In Sect. 3.1.3, we study the general case.

3.1.1 Torsion sheaves supported in a smooth hypersurface Let Z be a smooth hypersurface of X where $\dim X \leq n$. We define $G_Z(X)$ as the Grothendieck group of the category of coherent sheaves on X supported by Z. By dévissage, there is an isomorphism $i_{Z*} : G(Z) \xrightarrow{\sim} G_Z(X)$ (see [23]). For $\mathcal{G}$ coherent on Z, we define $\mathrm{ch}\left(i_{Z*}\mathcal{G}\right) = i_{Z*}\left(\mathrm{ch}(\mathcal{G})\,\mathrm{td}(N_{Z/X})^{-1}\right)$, where $\mathrm{ch}(\mathcal{G})$ is defined by induction. If $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of coherent sheaves on Z, by (W_{n-1}), we have $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{G}') + \mathrm{ch}(\mathcal{G}'')$. Thus $\mathrm{ch}\left(i_{Z*}\mathcal{G}\right) = \mathrm{ch}\left(i_{Z*}\mathcal{G}'\right) + \mathrm{ch}\left(i_{Z*}\mathcal{G}''\right)$. The map $\mathcal{G} \mapsto \mathrm{ch}(i_{Z*}\mathcal{G})$ factors over $i_{Z*}(G(Z)) \simeq G_Z(X)$. The resulting morphism from $G_Z(X)$ to $A(X)$ will be denoted by ch_Z.

The assertions of the following proposition are particular cases of (C_n), (F_n), and (P_n).

Proposition 3 *Let Z be a smooth hypersurface of X.*

(i) $\forall x \in G_Z(X)$, $\mathrm{ch}\left(i_Z^! x\right) = i_Z^* \mathrm{ch}_Z(x)$.
(ii) *If $\mathcal{E}$ is locally free on X and $x \in G_Z(X)$, then* $\mathrm{ch}_Z([\mathcal{E}].\,x) = \overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}_Z(x)$.

Proof (i) We can write $x = i_{Z*}\overline{x}$. Then,

$$\begin{aligned}
i_Z^* \mathrm{ch}_Z(x) &= i_Z^* i_{Z*}\left(\mathrm{ch}(\overline{x})\,\mathrm{td}\left(N_{Z/X}\right)^{-1}\right) = \mathrm{ch}(\overline{x})\,\mathrm{td}\left(N_{Z/X}\right)^{-1} c_1\left(N_{Z/X}\right)\\
&= \mathrm{ch}(\overline{x})\left[1 - e^{-c_1\left(N_{Z/X}\right)}\right]\\
&= \mathrm{ch}(\overline{x})\,\mathrm{ch}\left(i_Z^! i_{Z*}\mathcal{O}_Z\right) = \mathrm{ch}\left(\overline{x}\,.\,i_Z^! i_{Z*}\mathcal{O}_Z\right)\\
&= \mathrm{ch}\left(i_Z^! x\right), \quad \text{by Axiom B (vi), (C}_{n-1}\text{) and (P}_{n-1}\text{)}.
\end{aligned}$$

(ii) We have

$$\begin{aligned}
\mathrm{ch}_Z\left([\mathcal{E}]\,.\,x\right) &= \mathrm{ch}_Z\left(i_{Z*}\left(i_Z^![\mathcal{E}]\,.\,\overline{x}\right)\right) = i_{Z*}\left(\mathrm{ch}\left(i_Z^!\,[\mathcal{E}]\,.\,\overline{x}\right)\mathrm{td}\left(N_{Z/X}\right)^{-1}\right)\\
&= i_{Z*}\left(i_Z^*\,\overline{\mathrm{ch}}\,(\mathcal{E})\,\mathrm{ch}(\overline{x})\,\mathrm{td}\left(N_{Z/X}\right)^{-1}\right)\\
&= \overline{\mathrm{ch}}\,(\mathcal{E})\;i_{Z*}\left(\mathrm{ch}(\overline{x})\,\mathrm{td}\left(N_{Z/X}\right)^{-1}\right)\\
&= \overline{\mathrm{ch}}\,(\mathcal{E})\,\mathrm{ch}_Z(x), \quad \text{by (P}_{n-1}\text{), (C}_{n-1}\text{) and the projection formula.}
\end{aligned}$$

□

3.1.2 Torsion sheaves supported in a simple normal crossing divisor Let D be a divisor in X with simple normal crossing. We have an exact sequence:

$$\bigoplus_{i<j} G_{D_{ij}}(X) \longrightarrow \bigoplus_i G_{D_i}(X) \longrightarrow G_D(X) \longrightarrow 0.$$

If $\mathcal{F} \in G(D_{ij})$, using (RR$_{n-1}$) and the multiplicativity of the Todd class, we get $\mathrm{ch}_{D_i}\left(i_{D_{ij}*}\mathcal{F}\right) = i_{D_i*}\big(\mathrm{ch}\left(i_{D_{ij}/D_i*}\mathcal{F}\right)\mathrm{td}\left(N_{D_i/X}\right)^{-1}\big) = i_{D_{ij}*}\big(\mathrm{ch}(\mathcal{F})\,\mathrm{td}\left(N_{D_{ij}/X}\right)^{-1}\big)$. Thus $\mathrm{ch}_{D_i}\left(i_{D_{ij}*}\mathcal{F}\right) = \mathrm{ch}_{D_j}\left(i_{D_{ij}*}\mathcal{F}\right)$, and the map $\bigoplus_i \mathrm{ch}_{D_i}$ induces a map ch_D from $G_D(X)$ to $A(X)$.

Proposition 4 *The classes* ch_D *have the following properties*:

(i) *If $\mathcal{E}$ is locally free on X and $x \in G_D(X)$, then* $\mathrm{ch}_D([\mathcal{E}]\,.\,x) = \overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}_D(x)$.
(ii) *Let $\widetilde{D}$ be an effective divisor in X such that $\widetilde{D}^{red} = D$. Then*

$$\mathrm{ch}_D\left(\mathcal{O}_{\widetilde{D}}\right) = 1 - \overline{\mathrm{ch}}\left(\mathcal{O}_X(-\widetilde{D})\right).$$

(iii) (First lemma of functoriality) *Let* $f : X \to Y$ *be a surjective map. Let* D *be a simple normal crossing divisor in* Y *such that* $f^{-1}(D)$ *is also a simple normal crossing divisor in* X*. Then* $\forall y \in G_D(Y)$, $\mathrm{ch}_{f^{-1}(D)}\left(f^{!}y\right) = f^{*}\,\mathrm{ch}_D(y)$.

(iv) (Second lemma of functoriality) *Let* Y *be a smooth submanifold of* X *and* D *be a simple normal crossing divisor in* X*. Then,* $\forall x \in G_D(X)$, $\mathrm{ch}\left(i_Y^{!}x\right) = i_Y^{*}\,\mathrm{ch}_D(x)$.

We start with two technical lemmas which will be crucial for the proof of (ii) and (iii).

Lemma 1 *Let* $D = m_1 D_1 + \cdots + m_N D_N$ *be an effective divisor in* X *such that* D^{red} *is a simple normal crossing divisor, and* $\mu \in A(X)$ *be defined by*

$$\mu = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k!} \left(m_1\{D_1\} + \cdots + m_N\{D_N\}\right)^{k-1}.$$

Then there exist u_i *in* $G_{D_i}(X)$, $1 \leq i \leq N$, *and* ζ_{ij} *in* $A(D_{ij})$, $1 \leq i, j \leq N$, $i \neq j$, *such that*

(a) $u_1 + \cdots + u_N = \mathcal{O}_D$ *in* $G_{D^{\mathrm{red}}}(X)$.

(b) $\zeta_{ij} = -\zeta_{ji}$, $1 \leq i, j \leq N$, $i \neq j$.

(c) $$\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\left(N_{D_i/X}\right)^{-1} - m_i\, i_{D_i}^{*}\, \mu = \sum_{\substack{j=1\\ j\neq i}}^{N} i_{D_{ij}/D_i *}\zeta_{ij}, \quad 1 \leq i \leq N,$$

where $i_{D_i *}\overline{u}_i = u_i$.

Proof We proceed by induction on the number N of irreducible components of D^{red}. If $N = 1$, we must prove that $\mathrm{ch}(\overline{u}_1)\,\mathrm{td}\left(N_{D_1/X}\right)^{-1} = m_1\, i_{D_1}^{*}\,\mu$, where $u_1 = \mathcal{O}_{m_1 D_1}$. In $G_{D_1}(X)$ we have $\mathcal{O}_{m_1 D_1} = \sum_{q=0}^{m_1-1} i_{D_1 *}\left(N_{D_1/X}^{*\otimes q}\right)$, thus $\overline{u}_1 = \sum_{q=0}^{m_1-1} N_{D_1/X}^{*\otimes q}$. Therefore

$$\mathrm{ch}(\overline{u}_1)\,\mathrm{td}\left(N_{D_1/X}\right)^{-1} = \left(\sum_{q=0}^{m_1-1} e^{-q\, c_1\left(N_{D_1/X}\right)}\right) \frac{1 - e^{-c_1\left(N_{D_1/X}\right)}}{c_1\left(N_{D_1/X}\right)}$$
$$= \frac{1 - e^{-m_1 c_1\left(N_{D_1/X}\right)}}{c_1\left(N_{D_1/X}\right)} = m_1\, i_{D_1}^{*}\,\mu\,.$$

Suppose that the lemma holds for divisors D' such that D'^{red} has $N-1$ irreducible components. Let $D = m_1 D_1 + \cdots + m_N D_N$ and $D' = m_1 D_1 + \cdots + m_{N-1} D_{N-1}$. By induction, there exist u_i' in $G_{D_i}(X)$, $1 \leq i \leq N-1$, and ζ_{ij}' in $A(D_{ij})$, $1 \leq i, j \leq N-1$, $i \neq j$, satisfying properties (a), (b), and (c) of Lemma 1. For $0 \leq k \leq m_N$, we introduce the divisors $Z_k = m_1 D_1 + \cdots + m_{N-1} D_{N-1} + k D_N$. We have exact sequences $0 \to i_{D_N}^{*}\mathcal{O}_X(-Z_k) \to \mathcal{O}_{Z_{k+1}} \to \mathcal{O}_{Z_k} \to 0$. Thus, in $G_{D^{\mathrm{red}}}(X)$, we have

$$
\begin{aligned}
\mathcal{O}_D &= \mathcal{O}_{D'} + i_{D_N *}\left[\sum_{q=0}^{m_{N-1}} i^*_{D_N}\mathcal{O}_X(-Z_q)\right]\\
&= \mathcal{O}_{D'} + i_{D_N *} i^*_{D_N}\left[\mathcal{O}_X(-D')\sum_{q=0}^{m_{N-1}}\mathcal{O}_X\left(-qD_N\right)\right].
\end{aligned}
$$

We choose

$$
\begin{cases}
u_i = u'_i \quad \text{for} \quad 1 \le i \le N-1\\
u_N = i_{D_N *} i^*_{D_N}\left[\mathcal{O}_X(-D')\displaystyle\sum_{q=0}^{m_{N-1}}\mathcal{O}_X\left(-qD_N\right)\right].
\end{cases}
$$

Let i be such that $1 \le i \le N-1$. Then, by induction,

$$
\begin{aligned}
\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\left(N_{D_i/X}\right) - m_i\, i^*_{D_i}\,\mu &= \mathrm{ch}(\overline{u}'_i)\,\mathrm{td}\left(N_{D_i/X}\right) - m_i\, i^*_{D_i}\,\mu' + m_i\, i^*_{D_i}\,(\mu'-\mu)\\
&= \sum_{\substack{j=1\\ j\neq i}}^{N-1} i_{D_{ij}\,/\,D_i *}\zeta'_{ij} + m_i\ i^*_{D_i}\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\right.\\
&\quad\left.\times\sum_{j=1}^{k-1}\binom{k-1}{j}\left(\sum_{r=1}^{N-1} m_r\{D_r\}\right)^{k-1-j}\left(m_N\{D_N\}\right)^j\right]\\
&= \sum_{\substack{j=1\\ j\neq i}}^{N-1} i_{D_{ij}\,/\,D_i *}\zeta'_{ij} + m_i\ i_{D_{iN}\,/\,D_i *} i^*_{D_{iN}}\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\right.\\
&\quad\left.\times\sum_{j=1}^{k-1}\binom{k-1}{j}\left(\sum_{r=1}^{N-1} m_r\{D_r\}\right)^{k-1-j} m_N^j\{D_N\}^{j-1}\right].
\end{aligned}
$$

For the last equality, we have used that if $\{D_{iN}\}$ is the cycle class of D_{iN} in D_i, then $i^*_{D_i}\left(\alpha\left\{D_N\right\}\right) = i^*_{D_i}\alpha\left\{D_{iN}\right\} = i_{D_{iN}/D_i *}\left(i^*_{D_{iN}}\alpha\right)$.

Let us define

$$
\begin{cases}
\zeta_{ij} = \zeta'_{ij} & \text{if } 1 \le i, j \le N-1, \quad i \neq j,\\
\zeta_{iN} = m_i\ i^*_{D_{iN}}\left[\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\left(\sum_{r=1}^{N-1} m_r\{D_r\}\right)^{k-1-j} m_N^j\{D_N\}^{j-1}\right] & \\
& \text{if } 1 \le i \le N-1,\\
\zeta_{Nj} = -\zeta_{jN} & \text{if } 1 \le j \le N-1.
\end{cases}
$$

Properties (a) and (b) of Lemma 1 hold, and property (c) of the same lemma holds for $1 \leq i \leq N-1$. For $i = N$, let us now compute both members of (c). We have

$$\sum_{l=1}^{N-1} i_{D_{Nl}\,/\,D_N *}\zeta_{Nl} = \sum_{l=1}^{N-1} m_l\, i_{D_N}^* \left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{j=1}^{k-1} \binom{k-1}{j} \times \left(\sum_{r=1}^{N-1} m_r\{D_r\}\right)^{k-1-j} m_N^j \{D_N\}^{j-1}\{D_l\}\right]$$
$$= i_{D_N}^* \left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sum_{j=1}^{k-1} \binom{k-1}{j} \times \left(\sum_{r=1}^{N-1} m_r\{D_r\}\right)^{k-j} m_N^j \{D_N\}^{j-1}\right]. \qquad (*)$$

In the first equality, we have used $i_{D_{Nl}\,/\,D_N *}\, i_{D_{lN}}^* \alpha = i_{D_N}^* \alpha\,\{D_{lN}\} = i_{D_N}^*\,(\alpha\,\{D_l\})$, where $\{D_{lN}\}$ is the cycle class of D_{lN} in D_N. Now, by (C_{n-1}),

$$\mathrm{ch}(\overline{u}_N)\,\mathrm{td}\left(N_{D_N/X}\right)^{-1} - m_N\, i_{D_N}^*\,\mu = i_{D_N}^*\left[e^{-\sum_{r=1}^{N-1} m_r\{D_r\}} \left(\sum_{q=0}^{m_{N-1}} e^{-q\{D_N\}}\right) \times \frac{1-e^{-\{D_N\}}}{\{D_N\}}\right] - m_N\, i_{D_N}^*\,\mu$$
$$= i_{D_N}^*\left[e^{-\sum_{s=1}^{N-1} m_s\{D_s\}}\ \frac{1-e^{-m_N\{D_N\}}}{\{D_N\}} - m_N\,\mu\right]$$
$$= i_{D_N}^*\left[m_N \sum_{r=0}^{\infty}\sum_{q=1}^{\infty} \frac{(-1)^{r+q-1}}{r!\,q!}\left(\sum_{s=1}^{N-1} m_s\{D_s\}\right)^r \times \left(m_N\{D_N\}\right)^{q-1} -m_N \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\sum_{j=0}^{k-1}\binom{k-1}{j}\left(\sum_{s=1}^{N-1} m_s\{D_s\}\right)^{k-1-j}\left(m_N\{D_N\}\right)^j\right].$$

In the first term, we put $k = q + r$, $p = q - 1$ and we obtain

$$m_N\, i_{D_N}^*\left[\sum_{k=1}^{\infty}\sum_{p=0}^{k-1}\frac{(-1)^{k-1}}{k!}\left(\binom{k}{p+1} - \binom{k-1}{p}\right) \times\left(\sum_{s=1}^{N-1} m_s\{D_s\}\right)^{k-1-p}\left(m_N\{D_N\}\right)^p\right]. \qquad (**)$$

Now $\binom{k}{p+1} - \binom{k-1}{p}$ is equal to $\binom{k-1}{p+1}$ for $p \leq k-2$ and to zero for $p = k-1$. It suffices to put $j = p+1$ in the sum to obtain the equality of (*) and (**). □

Lemma 2 *Using the same notations as in Lemma 1, let α_i in $A(D_i)$, $1 \leq i \leq N$, be such that $\forall\, i, j$, $1 \leq i, j \leq N$, $i^*_{D_{ij}/D_i}\alpha_i = i^*_{D_{ij}/D_j}\alpha_j$. Then there exist elements u_i in $G_{D_i}(X)$, satisfying $u_1 + \cdots + u_N = \mathcal{O}_D$ in $G_{D^{\mathrm{red}}}(X)$, such that*

$$\sum_{i=1}^{N} i_{D_i *}\left(\alpha_i \;\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\left(N_{D_i/X}\right)^{-1}\right) = \left(\sum_{i=1}^{N} m_i\, i_{D_i *}(\alpha_i)\right)\mu.$$

Proof We pick $u_1, \ldots, u_N$ given by Lemma 1. Then

$$\begin{aligned}
&\sum_{i=1}^{N} i_{D_i *}\left(\alpha_i \;\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\left(N_{D_i/X}\right)^{-1}\right) - \left(\sum_{i=1}^{N} m_i\, i_{D_i *}(\alpha_i)\right)\mu \\
&\quad = \sum_{i=1}^{N} i_{D_i *}\left[\alpha_i \left(\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\left(N_{D_i/X}\right)^{-1} - m_i\, i^*_{D_i}\,\mu\right)\right] \\
&\quad = \sum_{i=1}^{N} \sum_{\substack{j=1\\ j\neq i}}^{N} i_{D_i *}\left[\alpha_i\; i_{D_{ij}/D_i *}\zeta_{ij}\right] = \sum_{i=1}^{N} \sum_{\substack{j=1\\ j\neq i}}^{N} i_{D_{ij} *}\left(i^*_{D_{ij}/D_i}\alpha_i\;\zeta_{ij}\right)
\end{aligned}$$

by the projection formula. Putting together the terms (i, j) and (j, i), we get 0, since $\zeta_{ij} = -\zeta_{ji}$. □

Proof of Proposition 4 (i) We write $x = \sum_{i=1}^{N} x_i$ in $G_{D^{\mathrm{red}}}(X)$, $x_i \in G_{D_i}(X)$. Then $\mathrm{ch}_D([\mathcal{E}]\,.\,x) = \sum_{i=1}^{N}\mathrm{ch}_{D_i}([\mathcal{E}]\,.\,x_i) = \sum_{i=1}^{N}\overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}_{D_i}(x_i) = \overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}_D(x)$ by Proposition 3 (ii) and the very definition of $\mathrm{ch}_D(x)$.

(ii) We choose $u_1, \ldots, u_N$ such that Lemma 1 holds. Then we have

$$\begin{aligned}
\mathrm{ch}\left(\mathcal{O}_{\widetilde{D}}\right) &= \sum_{i=1}^{N}\mathrm{ch}(u_i) = \sum_{i=1}^{N} i_{D_i *}\left(\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\left(N_{\widetilde{D}_i/X}\right)^{-1}\right) = \left(\sum_{i=1}^{N} m_i\{\widetilde{D}_i\}\right)\mu \\
&= 1 - e^{-\left(\sum_{r=1}^{N} m_r\{\widetilde{D}_r\}\right)} = 1 - \overline{\mathrm{ch}}\left(\mathcal{O}_X(-\widetilde{D})\right).
\end{aligned}$$

(iii) By dévissage we can suppose that D is a smooth hypersurface of Y. Let $\overline{f}_i$ be defined by the diagram

$$\begin{array}{ccc}
\widetilde{D_i} & \longrightarrow & X \\
{\scriptstyle \overline{f}_i}\downarrow & & \downarrow{\scriptstyle f} \\
D & \longrightarrow & Y
\end{array}$$

and let $y \in G(D)$. We put $\alpha_i = \overline{f}_i^{\,*}\,\mathrm{ch}(y)$. By the functoriality property (F_{n-1}) we have $i^*_{\widetilde{D}_{ij}/\widetilde{D}_i}\alpha_i = i^*_{\widetilde{D}_{ij}/\widetilde{D}_j}\alpha_j$. We choose again $u_1, \ldots, u_N$ such that Lemma 1 holds. Using functoriality properties for analytic K-theory with support (see [15]), Lemma 1 (a) implies $f^{\,!}i_{D*}y = \sum_{i=1}^N i_{\widetilde{D}_i*}\left[(\overline{f}_i^{\,!}y)\,.\,\overline{u}_i\right]$. Thus

$$
\begin{aligned}
\mathrm{ch}_{\widetilde{D}}\left(f^{\,!}i_{D*}y\right) &= \sum_{i=1}^N i_{\widetilde{D}_i*}\left(\mathrm{ch}\left(\overline{f}_i^{\,!}y\right)\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\left(N_{\widetilde{D}_i/X}\right)^{-1}\right)\\
&= \sum_{i=1}^N i_{\widetilde{D}_i*}\left(\alpha_i\,\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\left(N_{\widetilde{D}_i/X}\right)^{-1}\right)\\
&= \left(\sum_{i=1}^N m_i\ i_{\widetilde{D}_i*}(\alpha_i)\right)\mu\\
&= \left[\sum_{i=1}^N m_i\ i_{\widetilde{D}_i*}\left(\overline{f}_i^{\,*}\,\mathrm{ch}(y)\right)\right] f^*\left(\frac{1-e^{-\{D\}}}{\{D\}}\right)\\
&= f^*\left[i_{D*}\,(\mathrm{ch}(y))\cdot\frac{1-e^{-\{D\}}}{\{D\}}\right]\\
&= f^* i_{D*}\left(\mathrm{ch}(y)\,\mathrm{td}\left(N_{D/Y}\right)^{-1}\right)\\
&= f^*\ \mathrm{ch}_D\left(i_{D*}y\right)
\end{aligned}
$$

by (P_{n-1}), (F_{n-1}), Lemma 2, Axiom B (iv) and the projection formula.

(iv) We will first prove it under the assumption that, for all i, either Y and D_i intersect transversally, or $Y = D_i$. By dévissage, we can suppose that D has only one irreducible component and that Y and D intersect transversally, or $Y = D$. We deal with both cases separately.

- If Y and D intersect transversally, $i_Y^!\left[i_{D*}\mathcal{O}_D\right] = \left[i_{Y\cap D/Y*}\mathcal{O}_{Y\cap D}\right]$. Thus, if $x = i_{D*}\overline{x}$, then $i_Y^! x = i_{Y\cap D/Y*}\left(i^!_{Y\cap D/D}\overline{x}\right)$. We obtain

$$
\begin{aligned}
\mathrm{ch}\left(i_Y^! x\right) &= i_{Y\cap D/Y*}\left(\mathrm{ch}\left(i^!_{Y\cap D/D}\overline{x}\right)\mathrm{td}\left(N_{Y\cap D/Y}\right)^{-1}\right)\\
&= i_{Y\cap D/Y*}\left(i^*_{Y\cap D/D}\,\mathrm{ch}(\overline{x})\ i^*_{Y\cap D/D}\,\mathrm{td}\left(N_{D/X}\right)^{-1}\right)\\
&= i_Y^*\,i_{D*}\left(\mathrm{ch}(\overline{x})\,\mathrm{td}\left(N_{D/X}\right)^{-1}\right) = i_Y^*\,\mathrm{ch}_D(x),
\end{aligned}
$$

by (RR_{n-1}), (F_{n-1}) and Axiom B (iii).

- If $Y = D$, $i_Y^! \left[i_{D*}\mathcal{O}_D \right] = \left[\mathcal{O}_Y \right] - \left[N^*_{Y/X} \right]$. Thus $i_Y^! \, x = \overline{x} - \overline{x} \, . \, \left[N^*_{Y/X} \right]$ and

$$\operatorname{ch}\left(i_Y^! x\right) = \operatorname{ch}(\overline{x}) - \operatorname{ch}(\overline{x})\overline{\operatorname{ch}}\left(N^*_{Y/X}\right) = \operatorname{ch}(\overline{x})\left[1 - e^{-c_1\left(N_{Y/X}\right)}\right]$$

$$= \operatorname{ch}(\overline{x}) \operatorname{td}\left(N_{Y/X}\right)^{-1} c_1\left(N_{Y/X}\right) = i_Y^* i_{Y*}\left(\operatorname{ch}(\overline{x}) \operatorname{td}\left(N_{Y/X}\right)^{-1}\right)$$

$$= i_Y^* \operatorname{ch}_Y(x),$$

by (P_{n-1}), (C_{n-1}) and Axiom B (vi).

We examine now the general case. By Hironaka's desingularization theorem [19], we can desingularize $Y \cup D$ by a succession τ of k blowups with smooth centers such that $\tau^{-1}(Y \cup D)$ is a divisor with simple normal crossing. By first blowing up X along Y, we can suppose that $\tau^{-1}(Y) = \check{D}$ is a subdivisor of $\widetilde{D} = \tau^{-1}(Y \cup D)$. We have the following diagram:

$$\begin{array}{ccc} \check{D}_j & \xrightarrow{\;i_{\check{D}_j}\;} & \widetilde{X} \\ {\scriptstyle q_j}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\ Y & \xrightarrow[\;i_Y\;]{} & X \end{array}$$

Then $q_j^* \operatorname{ch}\left(i_Y^! x\right) = \operatorname{ch}\left(q_j^! i_Y^! x\right) = \operatorname{ch}\left(i_{\check{D}_j}^! \tau^! x\right) = i_{\check{D}_j}^* \operatorname{ch}_{\widetilde{D}}\left(\tau^! x\right) = i_{\check{D}_j}^* \tau^* \operatorname{ch}_D(x) = q_j^* i_Y^* \operatorname{ch}_D(x)$, by ($F_{n-1}$) and the first lemma of functoriality.

We can now write q_j as $\delta \circ \mu_j$, where E is the exceptional divisor of the blowup of X along Y, $\delta \colon E \to Y$ is the canonical projection and $\mu_j \colon \check{D}_j \to E$ is the restriction of the last $k-1$ blowups to $\check{D}_j$. Write $\tau = \tau_k \circ \tau_{k-1} \circ \cdots \circ \tau_1$ where τ_i are the blowups. Let us define a sequence of divisors $\left(E_i\right)_{0 \leq i \leq k}$ by induction: $E_0 = E$, and E_{i+1} is the strict transform of E_i under τ_{i+1}. Since the E_i are smooth divisors, all the maps $\tau_{i+1} \colon E_{i+1} \to E_i$ are isomorphisms. There exists j such that $E_k = \check{D}_j$. We deduce that $\mu_j = \tau_{|\check{D}_j} \colon \check{D}_j \to D$ is an isomorphism. Since δ is the projection of the projective bundle $\mathbb{P}\left(N_{Y/X}\right) \to Y$, δ^* is injective. Thus $q_j^* = \mu_j^* \delta^*$ is injective and we get $\operatorname{ch}\left(i_Y^! x\right) = i_Y^* \operatorname{ch}_D(x)$. □

Now, we can clear up the problem of the dependence of $\operatorname{ch}_D(\mathcal{F})$ with respect to D.

Proposition 5 *If D_1 and D_2 are two divisors of X with simple normal crossing such that* $\operatorname{supp}\mathcal{F} \subseteq D_1$ *and* $\operatorname{supp}\mathcal{F} \subseteq D_2$, *then* $\operatorname{ch}_{D_1}(\mathcal{F}) = \operatorname{ch}_{D_2}(\mathcal{F})$.

Proof This property is clear if $D_1 \subseteq D_2$. We will reduce the general situation to this case. By Hironaka's theorem, there exists $\tau : \widetilde{X} \to X$ such that $\tau^{-1}(D_1 \cup D_2)$ is a divisor with simple normal crossing. Let $\widetilde{D}_1 = \tau^{-1} D_1$ and $\widetilde{D}_2 = \tau^{-1} D_2$. By the first lemma of functoriality, since $\widetilde{D}_1 \subseteq \widetilde{D}$, we have $\tau^* \operatorname{ch}_{D_1}(\mathcal{F}) = \operatorname{ch}_{\widetilde{D}_1}\left(\tau^{!}[\mathcal{F}]\right) = \operatorname{ch}_{\widetilde{D}}\left(\tau^{!}[\mathcal{F}]\right)$. The same property holds for D_2. The map τ is a succession of blowups, thus τ^* is injective and we get $\operatorname{ch}_{D_1}(\mathcal{F}) = \operatorname{ch}_{D_2}(\mathcal{F})$. □

Definition 1 If $\operatorname{supp}(\mathcal{F}) \subseteq D$ where D is a simple normal crossing divisor, $\operatorname{ch}(\mathcal{F})$ is defined as $\operatorname{ch}_D(\mathcal{F})$.

By Proposition 5, this definition makes sense.

3.1.3 Torsion sheaves: the general case We can now define $\operatorname{ch}(\mathcal{F})$ for an arbitrary coherent torsion sheaf. Let $\mathcal{F}$ be a torsion sheaf. We say that a succession of blowups with smooth centers $\tau : \widetilde{X} \to X$ is a desingularization of $\mathcal{F}$ if there exists a divisor with simple normal crossing D in $\widetilde{X}$ such that $\tau^{-1}(\operatorname{supp}(\mathcal{F})) \subseteq D$. In that case, $\operatorname{ch}(\tau^{!}[\mathcal{F}])$ is defined by Definition 1. By Hironaka's theorem applied to $\operatorname{supp}(\mathcal{F})$, there always exists such a τ. We say that $\mathcal{F}$ can be desingularized in d steps if there exists a desingularization τ of $\mathcal{F}$ consisting of at most d blowups.

Proposition 6 *There exists a unique class* $\operatorname{ch}(\mathcal{F})$ *in* $A(X)$ *such that*

(i) *If* τ *is a desingularization of* $\mathcal{F}$, *then* $\tau^* \operatorname{ch}(\mathcal{F}) = \operatorname{ch}(\tau^{!}[\mathcal{F}])$.
(ii) *If* Y *is a smooth submanifold of* X, *then* $\operatorname{ch}\left(i_Y^{!}[\mathcal{F}]\right) = i_Y^* \operatorname{ch}(\mathcal{F})$.

Proof If τ is a desingularization of $\mathcal{F}$, then τ^* is injective by Axiom B (v). This proves that a class $\operatorname{ch}(\mathcal{F})$ satisfying (i) is unique. Let d be the number of blowups necessary to desingularize $\mathcal{F}$. Assertions (i) and (ii) will be proved at the same time by induction on d.

If $d = 0$, $\operatorname{supp}(\mathcal{F})$ is contained in a divisor with simple normal crossing D. Then properties (i) and (ii) are immediate consequences of the two lemmas of functoriality of Proposition 4.

Suppose now that Proposition 6 is proved for torsion sheaves which can be desingularized in $d - 1$ steps. Let $\mathcal{F}$ be a torsion sheaf which can be desingularized with at most d blowups. Let $(\widetilde{X}, \tau)$ be such a desingularization. We write τ as $\widetilde{\tau} \circ \tau_1$, where $\widetilde{\tau}$ is the first blowup in τ with E as exceptional divisor, as shown in the following diagram:

$$\begin{array}{ccc} & & \widetilde{X} \\ & & \Big\downarrow{\scriptstyle \tau_1} \\ E & \xrightarrow{\ i_E\ } & \widetilde{X}_1 \\ {\scriptstyle q}\Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{\tau}} \\ Y & \xrightarrow[\ i_Y\ ]{} & X \end{array}$$

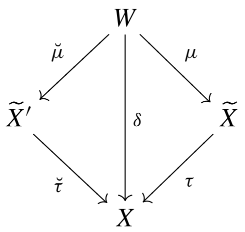
Then τ_1 consists of at most $d-1$ blowups and is a desingularization of the sheaves $\mathrm{Tor}_j(\mathcal{F}, \widetilde{\tau})$, $0 \le j \le n$. By induction, we can consider

$$\gamma\left(\widetilde{X}_1, \mathcal{F}\right) = \sum_{j=0}^{n} (-1)^j \,\mathrm{ch}\left(\mathrm{Tor}_j(\mathcal{F}, \widetilde{\tau})\right).$$

Now $i_E^*\gamma\left(\widetilde{X}_1, \mathcal{F}\right) = \sum_{j=0}^{n} (-1)^j \,\mathrm{ch}\left(i_E^!\left[\mathrm{Tor}_j(\mathcal{F}, \widetilde{\tau})\right]\right) = \mathrm{ch}\left(i_E^!\widetilde{\tau}^![\mathcal{F}]\right) = \mathrm{ch}\left(q^!i_Y^![\mathcal{F}]\right) = q^*\,\mathrm{ch}\left(i_Y^![\mathcal{F}]\right)$, by induction property (ii), (W_{n-1}) and (F_{n-1}). By Axiom B (v), there exists a unique class $\mathrm{ch}(\mathcal{F}, \tau)$ on X such that $\gamma\left(\widetilde{X}_1, \mathcal{F}\right) = \widetilde{\tau}^*\,\mathrm{ch}\,(\mathcal{F}, \tau)$.

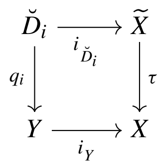
Now $\tau^*\,\mathrm{ch}(\mathcal{F}, \tau) = \tau_1^*\gamma\left(\widetilde{X}_1, \mathcal{F}\right) = \sum_{j=0}^{n} (-1)^j \,\mathrm{ch}\left(\tau_1^!\left[\mathrm{Tor}_j(\mathcal{F}, \widetilde{\tau})\right]\right) = \mathrm{ch}\left(\tau_1^!\widetilde{\tau}^![\mathcal{F}]\right) = \mathrm{ch}\left(\tau^![\mathcal{F}]\right)$, by induction property (i).

We can now prove (i). Let $\breve{\tau} : \widetilde{X}' \to X$ be an arbitrary desingularization of $\mathcal{F}$. We dominate the two resolutions τ, $\breve{\tau}$ by a third one, according to the diagram



Then $\breve{\mu}^*\,\mathrm{ch}\left(\breve{\tau}^![\mathcal{F}]\right) = \mathrm{ch}\left(\delta^![\mathcal{F}]\right) = \mu^*\mathrm{ch}\left(\tau^![\mathcal{F}]\right) = \mu^*\tau^*\,\mathrm{ch}(\mathcal{F}, \tau) = \breve{\mu}^*\breve{\tau}^*\mathrm{ch}(\mathcal{F}, \tau)$, by the first lemma of functoriality.

It remains to show (ii). For this, we desingularize $\mathrm{supp}(\mathcal{F}) \cup Y$ exactly as in the proof of the second lemma of functoriality. We have a diagram



where q_i^* is injective for at least one i. Then we obtain $q_i^*\left(i_Y^*\,\mathrm{ch}(\mathcal{F})\right) = i_{\breve{D}_i}^*\,\tau^*\,\mathrm{ch}(\mathcal{F}) = i_{\breve{D}_i}^*\,\mathrm{ch}(\tau^![\mathcal{F}]) = \mathrm{ch}\left(i_{\breve{D}_i}^!\,\tau^![\mathcal{F}]\right) = \mathrm{ch}\left(q_i^!\,i_Y^![\mathcal{F}]\right) = q_i^*\,\mathrm{ch}\left(i_Y^![\mathcal{F}]\right)$, by (i), the second lemma of functoriality and (F_{n-1}). Thus $i_Y^*\,\mathrm{ch}(\mathcal{F}) = \mathrm{ch}\left(i_Y^!\,[\mathcal{F}]\right)$. □

We have now completed the existence part of Theorem 3 for torsion sheaves.

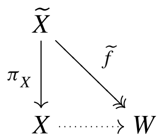
We turn to the proof of Proposition 2. So doing, we establish almost all the properties listed in the induction hypotheses for torsion sheaves.

Proof of Proposition 2 (i) Let $(\widetilde{X}, \tau)$ be a desingularization of $\mathrm{supp}(\mathcal{F}) \cup \mathrm{supp}(\mathcal{H})$ and D be the associated simple normal crossing divisor. Then $\tau^!\mathcal{F}$, $\tau^!\mathcal{G}$, $\tau^!\mathcal{H} \in$

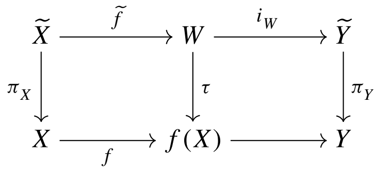
$G_D(\widetilde{X})$ and $\tau^!\mathcal{F} + \tau^!\mathcal{H} = \tau^!\mathcal{G}$ in $G_D(X)$. Thus, by Proposition 6 (i), $\tau^*[\mathrm{ch}(\mathcal{F})+\mathrm{ch}(\mathcal{H})] = \mathrm{ch}\left(\tau^![\mathcal{F}]\right) + \mathrm{ch}\left(\tau^![\mathcal{H}]\right) = \mathrm{ch}\left(\tau^![\mathcal{G}]\right) = \tau^*\mathrm{ch}(\mathcal{G})$. The map τ^* being injective, we get the Whitney formula for torsion sheaves.

(ii) The method is the same: let $x = [\mathcal{G}]$ and let τ be a desingularization of $\mathcal{G}$. Then, by Proposition 6 (i) and Proposition 4 (i), $\tau^*\mathrm{ch}([\mathcal{E}]\,.\,[\mathcal{G}]) = \mathrm{ch}\left(\tau^![\mathcal{E}]\,.\,\tau^![\mathcal{G}]\right) = \overline{\mathrm{ch}}\left(\tau^![\mathcal{E}]\right)\,.\,\mathrm{ch}\left(\tau^![\mathcal{G}]\right) = \tau^*\left(\overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}(\mathcal{G})\right)$.

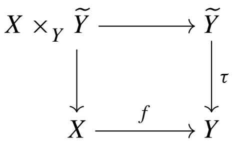
(iii) This property is known when f is the immersion of a smooth submanifold and when f is a bimeromorphic morphism by Proposition 6. Let us consider now the general case. By Grauert's direct image theorem, $f(X)$ is an irreducible analytic subset of Y. We desingularize $f(X)$ as an abstract complex space. We get a connected smooth manifold W and a bimeromorphic morphism $\tau : W \to f(X)$ obtained as a succession of blowups with smooth centers in $f(X)$. We perform a similar sequence of blowups, starting from $Y_1 = Y$ and blowing up at each step in Y_i the smooth center blown up at the ith step of the desingularization of $f(X)$. Let $\pi_Y : \widetilde{Y} \to Y$ be this morphism. The strict transform of $f(X)$ is W. The map $\tau : \tau^{-1}\left(f(X)_{\mathrm{reg}}\right) \xrightarrow{\sim} f(X)_{\mathrm{reg}}$ is an isomorphism. So we get a morphism $f(X)_{\mathrm{reg}} \to W$ which is in fact a meromorphic map from $f(X)$ to W, and finally, after composition on the left by f, from X to W. We desingularize this morphism:



and we get the following global diagram, where π_X is a bimeromorphic map:



Now $f \circ \pi_X = \pi_Y \circ \left(i_W \circ \widetilde{f}\right)$, and we know the functoriality formula for π_X, π_Y and i_W by Proposition 6. Since π_X^* is injective, it is enough to show the functoriality formula for $\widetilde{f}$. So we will assume that f is onto. Let $(\tau, \widetilde{Y})$ be a desingularization of $\mathcal{F}$. We have the diagram



where $\tilde{\tau}^{-1}(\operatorname{supp}\mathcal{F}) = D \subseteq \widetilde{Y}$ is a divisor with simple normal crossing and the map $X \times_Y \widetilde{Y} \to X$ is a bimeromorphic morphism. We have a meromorphic map $X \dashrightarrow X \times_Y \widetilde{Y}$, and we desingularize it by a morphism $T \to X \times_Y \widetilde{Y}$. Then we obtain the following commutative diagram, where $\pi : T \to X$ is a bimeromorphic map:

$$\begin{array}{ccc} T & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow{f} & Y \end{array}$$

Therefore, we can assume that $\operatorname{supp}(\mathcal{F})$ is included in a divisor with simple normal crossing D. We desingularize $f^{-1}(D)$ so that we are led to the case $\operatorname{supp}(\mathcal{F}) \subseteq D$, where D and $f^{-1}(D)$ are divisors with simple normal crossing in Y and X, respectively. In this case, we can use the first lemma of functoriality. □

3.2 A dévissage theorem for coherent sheaves

Let X be a complex compact manifold and $\mathcal{F}$ an analytic coherent sheaf on X. We have seen in Sect. 3.1 how to define $\operatorname{ch}(\mathcal{F})$ when $\mathcal{F}$ is a torsion sheaf.

Suppose that $\mathcal{F}$ has strictly positive generic rank. When $\mathcal{F}$ admits a global locally free resolution, we could try to define $\operatorname{ch}(\mathcal{F})$ the usual way. As explained in the introduction, this condition on $\mathcal{F}$ is not necessarily fulfilled. Even if such a resolution exists, the definition of $\operatorname{ch}(\mathcal{F})$ depends a priori on this resolution. A good substitute for a locally free resolution is a locally free quotient with maximal rank, since the kernel is then a torsion sheaf. Let $\mathcal{F}_{\mathrm{tor}} \subseteq \mathcal{F}$ be the maximal torsion subsheaf of $\mathcal{F}$. Then $\mathcal{F}$ admits a locally free quotient $\mathcal{E}$ of maximal rank if and only if $\mathcal{F} \big/ \mathcal{F}_{\mathrm{tor}}$ is locally free. In this case, $\mathcal{E} \simeq \mathcal{F} \big/ \mathcal{F}_{\mathrm{tor}}$ and we will say that $\mathcal{F}$ is *locally free modulo torsion*.

Unfortunately, such a quotient does not exist in general (for instance, take a torsion-free sheaf which is not locally free), but it exists up to a bimeromorphic morphism.

Proposition 7 *Let X be a complex compact manifold and $\mathcal{F}$ a coherent analytic sheaf on X. There exists a bimeromorphic morphism $\sigma : \widetilde{X} \to X$, which is a finite composition of blowups with smooth centers, such that $\sigma^*\mathcal{F}$ is locally free modulo torsion.*

Proof This is an immediate consequence of Hironaka's flattening theorem (see [20] and in the algebraic case [18]). □

3.3 Construction of the classes in the general case

Let X be a complex compact manifold of dimension n.

3.3.1 The case of sheaves which are locally free modulo torsion Let $\mathcal{F}$ be a coherent sheaf on X which is locally free modulo torsion. We define $\operatorname{ch}(\mathcal{F})$ as $\operatorname{ch}(\mathcal{F}_{\mathrm{tor}}) + \overline{\operatorname{ch}}(\mathcal{F}/\mathcal{F}_{\mathrm{tor}})$, where $\operatorname{ch}(\mathcal{F}_{\mathrm{tor}})$ has been constructed in Sect. 3.1.

We state now the Whitney formulae which apply to the Chern characters we have defined above.

Proposition 8 *Let* $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ *be an exact sequence of coherent analytic sheaves on* X. *Then* $\mathrm{ch}(\mathcal{F})$, $\mathrm{ch}(\mathcal{G})$ *and* $\mathrm{ch}(\mathcal{H})$ *have been previously defined and verify* $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$ *under any of the following hypotheses*:

(i) $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ *are locally free sheaves on* X.
(ii) $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ *are torsion sheaves.*
(iii) $\mathcal{G}$ *is locally free modulo torsion and* $\mathcal{F}$ *is a torsion sheaf.*

Proof (i) This is the usual theory for locally free sheaves.
(ii) This is Proposition 2 (i).
(iii) We have an exact sequence $0 \to \mathcal{T} \to \mathcal{G} \to \mathcal{E} \to 0$ where $\mathcal{T}$ is a torsion sheaf and $\mathcal{E}$ is locally free. Since $\mathcal{F}$ is a torsion sheaf, the morphism $\mathcal{F} \to \mathcal{G} \to \mathcal{E}$ is identically zero. Let us define $\mathcal{T}'$ by the exact sequence $0 \to \mathcal{T}' \to \mathcal{H} \to \mathcal{E} \to 0$. Then $\mathcal{T}'$ is a torsion sheaf which fits into the exact sequence $0 \to \mathcal{F} \to \mathcal{T} \longrightarrow \mathcal{T}' \to 0$. Thus $\mathcal{H}$ is locally free modulo torsion, so that $\mathrm{ch}(\mathcal{H})$ is defined, and $\mathrm{ch}(\mathcal{H}) = \overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}(\mathcal{T}') = \overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}(\mathcal{T}) - \mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{G}) - \mathrm{ch}(\mathcal{F})$ by (ii). □

Let us now look at the functoriality properties with respect to pullbacks.

Proposition 9 *Let* $f : X \to Y$ *be a holomorphic map. We assume that*

- $\dim Y = n$ *and* $\dim X \leq n$,
- *if* $\dim X = n$, f *is surjective.*

Then for every coherent sheaf on Y *which is locally free modulo torsion, the following properties hold*:

(i) *The Chern characters* $\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F}, f))$ *have been previously defined.*
(ii) $f^*\mathrm{ch}(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \,\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F}, f))$.

Proof (i) If $\dim X < n$, the classes $\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F}, f))$ are defined by induction. If $\dim X = n$ and f is surjective, then f is generically finite. Thus all the sheaves $\mathrm{Tor}_i(\mathcal{F}, f)$, $i \geq 1$, are torsion sheaves on X, so their Chern classes are defined by Proposition 2. The sheaf $f^*\mathcal{F}$ is locally free modulo torsion on X, so that $\mathrm{ch}(f^*\mathcal{F})$ is defined.

(ii) We have an exact sequence $0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{E} \to 0$ where $\mathcal{T}$ is a torsion sheaf and $\mathcal{E}$ is a locally free sheaf. Remark that, for $i \geq 1$, $\mathrm{Tor}_i(\mathcal{F}, f) \simeq \mathrm{Tor}_i(\mathcal{T}, f)$. Thus, by Proposition 2 (iii),

$$\begin{aligned}
&\sum_{i \geq 0} (-1)^i \,\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F}, f)) \\
&\quad = \overline{\mathrm{ch}}\left(f^*\mathcal{E}\right) + \mathrm{ch}\left(f^*\mathcal{T}\right) + \sum_{i \geq 1} (-1)^i \,\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{T}, f)) \\
&\quad = f^*\overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}\left(f^![\mathcal{T}]\right) = f^*\left(\overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}(\mathcal{T})\right) = f^*\mathrm{ch}(\mathcal{F}).
\end{aligned}$$

□

3.3.2 The general case We consider now an arbitrary coherent sheaf $\mathcal{F}$ on X. By Proposition 7, there exists $\sigma : \widetilde{X} \to X$ obtained as a finite composition of blowups with smooth centers such that $\sigma^* \mathcal{F}$ is locally free modulo torsion.

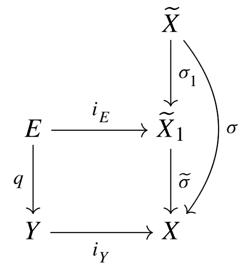
Proposition 10 *There exists a unique class* $\mathrm{ch}(\mathcal{F})$ *in* $A(X)$ *such that*

(i) *If* $\sigma : \widetilde{X} \to X$ *is a succession of blowups with smooth centers such that* $\sigma^* \mathcal{F}$ *is locally free modulo torsion, then* $\sigma^* \,\mathrm{ch}(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \,\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F}, \sigma))$.
(ii) *If* Y *is a smooth submanifold of* X, $\mathrm{ch}\left(i_Y^{!}[\mathcal{F}]\right) = i_Y^* \,\mathrm{ch}(\mathcal{F})$.

Remark 2 By Proposition 9 (i), all the terms in (i) are defined.

Proof By Axiom B (v), a class satisfying (i) is unique. We will prove the result by induction on the number d of blowups in σ as in Proposition 6. If $d = 0$, $\mathcal{F}$ is locally free modulo torsion and we can use Proposition 9.

Suppose now that (i) and (ii) hold at step $d - 1$. As usual, we look at the first blowup in σ:



The sheaves $\mathrm{Tor}_j(\mathcal{F}, \sigma)$ are torsion sheaves for $j \geq 1$ and $\sigma_1^* \,\mathrm{Tor}_0(\mathcal{F}, \widetilde{\sigma}) = \sigma^* \mathcal{F}$ is locally free modulo torsion. Since σ_1 consists of $d - 1$ blowups, we can define by induction on $\widetilde{X}_1$ the class $\gamma(\mathcal{F}) = \sum_{j \geq 0} (-1)^j \,\mathrm{ch}\left(\mathrm{Tor}_j(\mathcal{F}, \widetilde{\sigma})\right)$.

Lemma 3 $\sigma_1^* \gamma(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \,\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F}, \sigma))$.

Proof By induction,

$$\sigma_1^* \gamma(\mathcal{F}) = \sum_{p,q \geq 0} (-1)^{p+q} \,\mathrm{ch}\left[\mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F}, \widetilde{\sigma}), \sigma_1)\right] = \sum_{p,q \geq 0} (-1)^{p+q} \,\mathrm{ch}(E_2^{p,q})$$

where the Tor spectral sequence satisfies $E_2^{p,q} = \mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F}, \widetilde{\sigma}), \sigma_1)$ and $E_\infty^{p,q} = \mathrm{Gr}^p \,\mathrm{Tor}_{p+q}(\mathcal{F}, \sigma)$. All the $E_r^{p,q}$, $2 \leq r \leq \infty$, are torsion sheaves except perhaps $E_r^{0,0}$. Since no arrow $d_r^{p,q}$ starts from or arrives at $E_r^{0,0}$, we have in $G_{\mathrm{tors}}(X)$

$$\sum_{\substack{p,q \\ p+q \geq 1}} (-1)^{p+q} [E_2^{p,q}] = \sum_{\substack{p,q \\ p+q \geq 1}} (-1)^{p+q} [E_\infty^{p,q}] = \sum_{i \geq 1} (-1)^i \,[\mathrm{Tor}_i(\mathcal{F}, \sigma)]\,.$$

Using Proposition 8 (ii), we get

$$\sigma_1^{\;*}\gamma(\mathcal{F}) = \mathrm{ch}\left(E_2^{0,0}\right) + \mathrm{ch}\left(\sum_{i\geq 1}(-1)^i\,\mathrm{Tor}_i(\mathcal{F},\sigma)\right) = \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F},\sigma))\,.$$

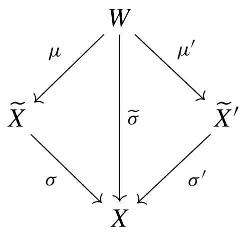
□

We compute

$$\begin{aligned} i_E^*\gamma(\mathcal{F}) &= i_E^*\left(\sum_{j\geq 0}(-1)^j\,\mathrm{ch}\left(\mathrm{Tor}_j(\mathcal{F},\sigma)\right)\right) = \sum_{j\geq 0}(-1)^j\,\mathrm{ch}\left(i_E^!\left[\mathrm{Tor}_j(\mathcal{F},\sigma)\right]\right) \\ &= \mathrm{ch}\left(i_E^!\widetilde{\sigma}^{\,!}[\mathcal{F}]\right) = \mathrm{ch}\left(q^{\,!}i_Y^![\mathcal{F}]\right) = q^*\,\mathrm{ch}\left(i_Y^![\mathcal{F}]\right), \end{aligned}$$

by induction property (ii) and (F_{n-1}). By Axiom B (v), there exists a unique class $\mathrm{ch}(\mathcal{F},\sigma)$ on X such that $\gamma(\mathcal{F}) = \widetilde{\sigma}^*\,\mathrm{ch}(\mathcal{F},\sigma)$. By Lemma 3, $\sigma^*\,\mathrm{ch}(\mathcal{F},\sigma) = \sigma_1^{\;*}\gamma(\mathcal{F}) = \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F},\sigma))$.

Let $\sigma'\colon \widetilde{X}'\to X$ be a succession of blowups with smooth centers such that $\sigma^*\mathcal{F}$ is locally free modulo torsion. We dominate the two resolutions σ and σ' by a third one as shown in the following diagram



Now, by Proposition 9 (ii) and Lemma 3,

$$\begin{aligned} \mu'^*\sigma'^*\,\mathrm{ch}(\mathcal{F},\sigma) &= \mu^*\sigma^*\,\mathrm{ch}(\mathcal{F},\sigma) = \mu^*\sum_{i\geq 0}(-1)^i\,\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F},\sigma)) \\ &= \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F},\widetilde{\sigma})) = \mu'^*\sum_{i\geq 0}(-1)^i\,\mathrm{ch}\left(\mathrm{Tor}_i(\mathcal{F},\sigma')\right), \end{aligned}$$

so that $\sigma'^*\,\mathrm{ch}(\mathcal{F},\sigma) = \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\left(\mathrm{Tor}_i(\mathcal{F},\sigma')\right)$.

We must now prove Proposition 10 (ii). Let Y be a smooth submanifold of X. We choose $\sigma : \widetilde{X} \to X$ such that $\sigma^* \mathcal{F}$ is locally free modulo torsion and $\sigma^{-1}(Y)$ is a simple normal crossing divisor with irreducible components D_j. Let j be such that q_j^* is injective, q_j being defined by the diagram

$$\begin{array}{ccc} D_j & \xrightarrow{\ i_{D_j}\ } & \widetilde{X} \\ {\scriptstyle q_j}\downarrow & & \downarrow{\scriptstyle \sigma} \\ Y & \xrightarrow{\ i_Y\ } & X \end{array}$$

We have $q_j^* \operatorname{ch}\left(i_Y^! [\mathcal{F}]\right) = \operatorname{ch}\left(q_j^! i_Y^! [\mathcal{F}]\right) = \operatorname{ch}\left(i_{D_j}^! \sigma^! [\mathcal{F}]\right) = \sum_{i \geq 0} (-1)^i \; i_{D_j}^* \operatorname{ch}\left(\operatorname{Tor}_i(\mathcal{F}, \sigma)\right)$ by Proposition 9 (ii). Now, by the point (i), we have $\sum_{i \geq 0} (-1)^i \operatorname{ch}\left(\operatorname{Tor}_i(\mathcal{F}, \sigma)\right) = \sigma^* \operatorname{ch}(\mathcal{F})$. Thus we get $q_j^* \operatorname{ch}(i_Y^! [\mathcal{F}]) = i_{D_j}^* \sigma^* \operatorname{ch}(\mathcal{F}) = q_j^* \left(i_Y^* \operatorname{ch}(\mathcal{F})\right)$. Therefore, we obtain $\operatorname{ch}\left(i_Y^! [\mathcal{F}]\right) = i_Y^* \operatorname{ch}(\mathcal{F})$ and the proof is complete. □

4 The Whitney formula

In the previous section, we achieved an important step in the induction process by defining the classes $\operatorname{ch}(\mathcal{F})$ when $\mathcal{F}$ is any coherent sheaf on a n-dimensional manifold. To conclude the proof of Theorem 3, it remains to check properties (W_n), (F_n) and (P_n). The crux of the proof is in fact property (W_n). The main result of this section is Theorem 4. The other induction hypotheses will be proved in Theorem 5.

Theorem 4 (W_n) *holds.*

To prove Theorem 4, we need several reduction steps.

4.1 Reduction to the case where $\mathcal{F}$ and $\mathcal{G}$ are locally free and $\mathcal{H}$ is a torsion sheaf

Proposition 11 *Suppose that* (W_n) *holds when* $\mathcal{F}$ *and* $\mathcal{G}$ *are locally free sheaves and* $\mathcal{H}$ *is a torsion sheaf. Then* (W_n) *holds for arbitrary coherent sheaves* $\mathcal{F}$, $\mathcal{G}$ *and* $\mathcal{H}$.

We start with a preliminary lemma:

Lemma 4 *It is sufficient to prove* (W_n) *when* $\mathcal{F}$, $\mathcal{G}$ *are locally free modulo torsion and* $\mathcal{H}$ *is a torsion sheaf.*

Proof We take a general exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$. Let $\sigma : \widetilde{X} \to X$ be a bimeromorphic morphism such that $\sigma^* \mathcal{F}$, $\sigma^* \mathcal{G}$ and $\sigma^* \mathcal{H}$ are locally free modulo torsion (we know that such a σ exists by Proposition 7). We have an exact sequence defining $\mathcal{Q}$ and $\mathcal{T}_1$:

$$\begin{array}{ccccccccccc}
 & & & & & & 0 \searrow \ \nearrow 0 & & & & \\
 & & & & & & \mathcal{Q} & & & & \\
 & & & & & & \nearrow \quad \searrow & & & & \\
\cdots \longrightarrow & \mathrm{Tor}_1(\mathcal{G},\sigma) & \longrightarrow & \mathrm{Tor}_1(\mathcal{H},\sigma) & \longrightarrow & \sigma^*\mathcal{F} & \longrightarrow & \sigma^*\mathcal{G} & \longrightarrow & \sigma^*\mathcal{H} & \longrightarrow 0 \\
 & & & & \searrow \quad \nearrow & & & & & & \\
 & & & & \mathcal{T}_1 & & & & & & \\
 & & & & 0 \nearrow \ \searrow 0 & & & & & &
\end{array}$$

Remark that $\mathcal{T}_1$ is a torsion sheaf. By Proposition 8 (iii), $\mathcal{Q}$ is locally free modulo torsion and $\mathrm{ch}(\sigma^*\mathcal{F}) = \mathrm{ch}(\mathcal{T}_1) + \mathrm{ch}(\mathcal{Q})$. Besides, $[\mathcal{T}_1] - \big[\mathrm{Tor}_1(\mathcal{H},\sigma)\big] + \big[\mathrm{Tor}_1(\mathcal{G},\sigma)\big] - \cdots = 0$ in $G_{\mathrm{tors}}(\widetilde{X})$. Then by Proposition 6 (i) and Proposition 8 (ii), $\sigma^*(\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G})) = \sum_{i\geq 0}(-1)^i\,[\mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{F},\sigma)) + \mathrm{ch}\,(\mathrm{Tor}_i(\mathcal{H},\sigma)) - \mathrm{ch}(\mathrm{Tor}_i(\mathcal{G},\sigma))] = \mathrm{ch}\,(\sigma^*\mathcal{F}) + \mathrm{ch}\,(\sigma^*\mathcal{H}) - \mathrm{ch}\,(\sigma^*\mathcal{G}) - \mathrm{ch}(\mathcal{T}_1) = \mathrm{ch}(\mathcal{Q}) + \mathrm{ch}\,(\sigma^*\mathcal{H}) - \mathrm{ch}\,(\sigma^*\mathcal{G})$. Since σ^* is injective, we can assume without loss of generality that $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are locally free modulo torsion. Let $\mathcal{E}_1$ be the locally free quotient of maximal rank of $\mathcal{H}$, so we have an exact sequence $0 \to \mathcal{T}_1 \to \mathcal{H} \to \mathcal{E}_1 \to 0$. We define $\mathcal{F}_1$ by the exact sequence $0 \to \mathcal{F}_1 \to \mathcal{G} \to \mathcal{E}_1 \to 0$. Then we get a third exact sequence $0 \to \mathcal{F} \to \mathcal{F}_1 \to \mathcal{T}_1 \to 0$. We have by definition $\mathrm{ch}(\mathcal{H}) = \overline{\mathrm{ch}}(\mathcal{E}_1) + \mathrm{ch}(\mathcal{T}_1)$. Thus, $\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G}) = (\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{T}_1) - \mathrm{ch}(\mathcal{F}_1)) + \big(\mathrm{ch}(\mathcal{F}_1) + \overline{\mathrm{ch}}(\mathcal{E}_1) - \mathrm{ch}(\mathcal{G})\big) - \big(\mathrm{ch}(\mathcal{T}_1) + \overline{\mathrm{ch}}(\mathcal{E}_1) - \mathrm{ch}(\mathcal{H})\big) = (\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{T}_1) - \mathrm{ch}(\mathcal{F}_1)) + \big(\mathrm{ch}(\mathcal{F}_1) + \overline{\mathrm{ch}}(\mathcal{E}_1) - \mathrm{ch}(\mathcal{G})\big)$. Let $\mathcal{E}_2$ be the locally free quotient of maximal rank of $\mathcal{G}$. We define $\mathcal{T}_2$ by the exact sequence $0 \to \mathcal{T}_2 \to \mathcal{G} \to \mathcal{E}_2 \to 0$. The morphism from $\mathcal{G}$ to $\mathcal{E}_1$ (via $\mathcal{H}$) induces a morphism $\mathcal{E}_2 \to \mathcal{E}_1$ which remains of course surjective. Let $\mathcal{E}_3$ be the kernel of this morphism, then $\mathcal{E}_3$ is a locally free sheaf. We get an exact sequence $0 \to \mathcal{T}_2 \to \mathcal{F}_1 \to \mathcal{E}_3 \to 0$. Therefore $\mathcal{F}_1$ is locally free modulo torsion and $\mathrm{ch}(\mathcal{F}_1) = \mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_3)$. On the other hand, $\overline{\mathrm{ch}}\,(\mathcal{E}_1) + \overline{\mathrm{ch}}\,(\mathcal{E}_3) = \overline{\mathrm{ch}}\,(\mathcal{E}_2)$ and we obtain $\mathrm{ch}\,(\mathcal{F}_1) + \overline{\mathrm{ch}}\,(\mathcal{E}_1) - \mathrm{ch}(\mathcal{G}) = \big(\mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_3)\big) + \big(\overline{\mathrm{ch}}(\mathcal{E}_2) - \overline{\mathrm{ch}}(\mathcal{E}_3)\big) - \big(\mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_2)\big) = 0$. Therefore, $\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{T}_1) - \mathrm{ch}(\mathcal{F}_1)$. Since $\mathcal{T}_1$ is a torsion sheaf, we are done. □

Proof of Proposition 11 By Lemma 4, we can suppose that $\mathcal{F}$, $\mathcal{G}$ are locally free modulo torsion and $\mathcal{H}$ is a torsion sheaf. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the locally free quotients of maximal rank of $\mathcal{F}$ and $\mathcal{G}$. The associated kernels will be denoted $\mathcal{T}_1$ and $\mathcal{T}_2$. The morphism $\mathcal{F} \to \mathcal{G}$ induces a morphism $\mathcal{T}_1 \to \mathcal{T}_2$. We get a morphism $\mathcal{E}_1 \to \mathcal{E}_2$ with torsion kernel and cokernel. Since $\mathcal{E}_1$ is a locally free sheaf, this morphism is injective.

In the following diagram, we introduce the cokernels $\mathcal{T}_3$ and $\mathcal{T}_4$:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E}_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}_2 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{E}_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathcal{T}_3 & & \mathcal{H} & & \mathcal{T}_4 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

By the snake lemma, $0 \to \mathcal{T}_3 \to \mathcal{H} \to \mathcal{T}_4 \to 0$ is an exact sequence of torsion sheaves. Then by Proposition 2 (i), $\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G}) = \mathrm{ch}\,(\mathcal{T}_1) + \overline{\mathrm{ch}}\,(\mathcal{E}_1) + \mathrm{ch}\,(\mathcal{T}_3) + \mathrm{ch}\,(\mathcal{T}_4) - \mathrm{ch}\,(\mathcal{T}_2) - \overline{\mathrm{ch}}\,(\mathcal{E}_2) = \overline{\mathrm{ch}}\,(\mathcal{E}_1) + \mathrm{ch}\,(\mathcal{T}_4) - \overline{\mathrm{ch}}\,(\mathcal{E}_2)$. This finishes the proof. □

4.2 A structure theorem for coherent torsion sheaves of projective dimension one

In Sect. 4.1 we have reduced the Whitney formula to the particular case where $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is a torsion sheaf. We are now going to prove that it is sufficient to suppose that $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersurface of X. The main tool of this section is the following proposition:

Proposition 12 *Let $\mathcal{H}$ be a torsion sheaf which admits a global locally free resolution of length two. Then there exist a bimeromorphic morphism $\sigma : \widetilde{X} \to X$ obtained by a finite number of blowups with smooth centers, an effective divisor D in X whose associated reduced divisor has simple normal crossing, and an increasing sequence $(D_i)_{1\le i\le r}$ of subdivisors of D such that $\sigma^*\mathcal{H}$ is everywhere locally isomorphic to $\bigoplus_{i=1}^r \mathcal{O}_{\widetilde{X}}/\mathcal{I}_{D_i}$.*

Proof Let $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{H} \to 0$ be a locally free resolution of $\mathcal{H}$, rank $(\mathcal{E}_1) =$ rank $(\mathcal{E}_2) = r$. Recall that the kth Fitting ideal of $\mathcal{H}$ is the coherent ideal sheaf generated by the determinants of all the $k \times k$ minors of M when M is any local matrix realization in coordinates of the morphism $\mathcal{E}_1 \to \mathcal{E}_2$ (see [8]). By Hironaka's theorem, we can suppose, after taking a finite number of pullbacks under blowups with smooth centers, that all the Fitting ideals $\mathrm{Fitt}_k(\mathcal{H})$ are ideal sheaves associated with effective divisors D'_k whose associated reduced divisors have simple normal crossing. Then it is easy to prove that $\mathcal{H}$ is everywhere locally isomorphic to $\bigoplus_{i=1}^r \mathcal{O}_{\widetilde{X}}/\mathcal{I}_{D_i}$, where $D_k = D'_k - D'_{k-1}$. □

From now on, we will say that a torsion sheaf $\mathcal{H}$ is *principal* if it is everywhere locally isomorphic to a fixed sheaf $\bigoplus_{i=1}^r \mathcal{O}_X/\mathcal{I}_{D_i}$ where the D_i are effective divisors such that D_i^{red} have simple normal crossing and $D_1 \leq D_2 \leq \cdots \leq D_r$. We will denote by $\nu(\mathcal{H})$ the number of irreducible components of D, counted with their multiplicities.

Proposition 13 *It suffices to prove the Whitney formula when $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersuface.*

Proof We proceed in several steps. By Proposition 11, it is enough to prove the Whitney formula when $\mathcal{F}$, $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is a torsion sheaf, so we suppose that $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ verify these hypotheses. By Proposition 12, there exists a bimeromorphic morphism $\sigma : \widetilde{X} \to X$ such that $\sigma^*\mathcal{H}$ is principal. The sheaf $\mathrm{Tor}_1(\mathcal{H}, \sigma)$ is zero since it is a torsion subsheaf of $\sigma^*\mathcal{F}$. Thus the sequence $0 \to \sigma^*\mathcal{F} \to \sigma^*\mathcal{G} \to \sigma^*\mathcal{H} \to 0$ is exact and $\sigma^*(\mathrm{ch}\,\overline{\mathcal{F}} + \mathrm{ch}\,\overline{\mathcal{G}} - \mathrm{ch}\,\mathcal{H}) = \overline{\mathrm{ch}}(\sigma^*\mathcal{F}) + \overline{\mathrm{ch}}(\sigma^*\mathcal{G}) = \mathrm{ch}(\sigma^*\mathcal{H})$ by Proposition 2 (iii). Then we argue by induction on $\nu(\mathcal{H})$. If $\nu(\mathcal{H}) = 0$, $\mathcal{H} = 0$ and $\mathcal{F} \simeq \mathcal{G}$. If $\nu(\mathcal{H}) = 1$, $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersurface and there is nothing to prove. In the general case, let Y be an irreducible component of D_1. Since $Y \leq D_i$ for every i with $1 \leq i \leq r$, $\mathcal{E} = \mathcal{H}_{|Y}$ is locally free on Y. If we define $\widetilde{\mathcal{H}}$ by the exact sequence $0 \to \widetilde{\mathcal{H}} \to \mathcal{H} \to i_{Y*}\mathcal{E} \to 0$, $\widetilde{\mathcal{H}}$ is everywhere locally isomorphic to $\bigoplus_{i=1}^{r} \mathcal{O}_X / \mathcal{I}_{D_i - Y}$. Thus $\widetilde{\mathcal{H}}$ is principal and $\nu(\widetilde{\mathcal{H}}) = \nu(\mathcal{H}) - 1$. We define the locally free sheaf $\widetilde{\mathcal{E}}$ by the exact sequence: $0 \to \widetilde{\mathcal{E}} \to \mathcal{G} \to i_{Y*}\mathcal{E} \to 0$. Furthermore, we have an exact sequence $0 \to \mathcal{F} \to \widetilde{\mathcal{E}} \to \widetilde{\mathcal{H}} \to 0$. By induction, $\overline{\mathrm{ch}}\left(\widetilde{\mathcal{E}}\right) = \overline{\mathrm{ch}}(\mathcal{F}) + \mathrm{ch}\left(\widetilde{\mathcal{H}}\right)$ and by our hypothesis $\mathrm{ch}(\mathcal{G}) = \overline{\mathrm{ch}}\left(\widetilde{\mathcal{E}}\right) + \mathrm{ch}\left(i_{Y*}\mathcal{E}\right)$. Since $\widetilde{\mathcal{H}}$, $\mathcal{H}$ and $i_{Y*}\mathcal{E}$ are torsion sheaves, $\mathrm{ch}(\mathcal{H}) = \mathrm{ch}\left(\widetilde{\mathcal{H}}\right) + \mathrm{ch}\left(i_{Y*}\mathcal{E}\right)$ and we get $\overline{\mathrm{ch}}(\mathcal{G}) = \overline{\mathrm{ch}}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$. This finishes the proof. □

4.3 Proof of the Whitney formula

We are now ready to prove Theorem 4. In the Sects. 4.1 and 4.2, we have made successive reductions in order to prove the Whitney formula in a tractable context, so that we are reduced to the case where $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves and $\mathcal{H} = i_{Y*}\mathcal{E}$, where Y is a smooth hypersurface of X and $\mathcal{E}$ is a locally free sheaf on Y. Our working hypotheses will be these.

Let us briefly explain the argument. We consider the sheaf $\widetilde{\mathcal{G}}$ on $X \times \mathbb{P}^1$ obtained by deformation of the second extension class of the exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$. Then $\widetilde{\mathcal{G}}_{|X\times\{0\}} \simeq \mathcal{F} \oplus \mathcal{H}$ and $\widetilde{\mathcal{G}}_{|X\times\{t\}} \simeq \mathcal{G}$ for $t \neq 0$. It will turn out that $\widetilde{\mathcal{G}}$ is locally free modulo torsion on the blowup of $X \times \mathbb{P}^1$ along $Y \times \{0\}$, and its torsion part $\mathcal{N}$ will be the push-forward of a locally free sheaf on the exceptional divisor E, say $\mathcal{N} = i_{E*}\mathcal{L}$. Then we consider the class $\alpha = \overline{\mathrm{ch}}(\mathcal{Q}) + i_{E*}\left(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}(N_{E/X})^{-1}\right)$ on the blowup, where $\mathcal{Q} = \widetilde{\mathcal{G}}/\mathcal{N}$. After explicit computations, it will appear that α is the pullback of a form β on the base $X \times \mathbb{P}^1$. By the $\mathbb{P}^1$-homotopy invariance of the cohomology theory (see Axiom A (iii)), $\beta_{|X\times\{t\}}$ does not depend on t. This will give the desired result.

Let us first introduce some notations. The morphism $\mathcal{F} \to \mathcal{G}$ will be denoted by γ. Let s be a global section of $\mathcal{O}_{\mathbb{P}^1}(1)$ which vanishes exactly at $\{0\}$. Let $\mathrm{pr}_1 : X \times \mathbb{P}^1 \to X$ be the projection on the first factor. The relative $\mathcal{O}(1)$, namely $\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$, will still be denoted by $\mathcal{O}(1)$. We define a sheaf $\widetilde{\mathcal{G}}$ on $X \times \mathbb{P}^1$ by the exact sequence $0 \to \mathrm{pr}_1^* \mathcal{F} \to \mathrm{pr}_1^* \mathcal{F}(1) \oplus \mathrm{pr}_1^* \mathcal{G} \to \widetilde{\mathcal{G}} \to 0$, where the first map is by $(\mathrm{id} \otimes s, \gamma)$. Remark that $\widetilde{\mathcal{G}}_0 \simeq \mathcal{F} \oplus \mathcal{H}$ and $\widetilde{\mathcal{G}}_t \simeq \mathcal{G}$ if $t \neq 0$.

Lemma 5 *There exist two exact sequences*

$$0 \longrightarrow \mathrm{pr}_1^* \mathcal{F}(1) \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{pr}_1^* \mathcal{H} \longrightarrow 0 \tag{i}$$

$$0 \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{pr}_1^* \mathcal{G}(1) \longrightarrow i_{X_0 *}\mathcal{H} \longrightarrow 0. \tag{ii}$$

Remark 3 (i) implies that $\widetilde{\mathcal{G}}$ is flat over $\mathbb{P}^1$.

Proof (i) The morphism $\mathrm{pr}_1^* \mathcal{F}(1) \oplus \mathrm{pr}_1^* \mathcal{G} \twoheadrightarrow \mathrm{pr}_1^* \mathcal{G} \twoheadrightarrow \mathrm{pr}_1^* \mathcal{H}$ induces a morphism $\widetilde{\mathcal{G}} \twoheadrightarrow \mathrm{pr}_1^* \mathcal{H}$. If $\mathcal{K}$ is the kernel of this morphism, the sequence

$$0 \longrightarrow \mathrm{pr}_1^* \mathcal{F} \longrightarrow \mathrm{pr}_1^* \mathcal{F}(1) \oplus \mathrm{pr}_1^* \mathcal{F} \longrightarrow \mathcal{K} \longrightarrow 0\ ,$$

where the first morphism is $(\mathrm{id} \otimes s,\ \mathrm{id})$, is exact. Thus $\mathcal{K} = \mathrm{pr}_1^* \mathcal{F}(1)$.

(ii) We consider the morphism

$$\mathrm{pr}_1^* \mathcal{F}(1) \oplus \mathrm{pr}_1^* \mathcal{G} \twoheadrightarrow \mathrm{pr}_1^* \mathcal{G}(1) \ \text{ defined by } \ f + g \longmapsto \gamma(f) - g \otimes s.$$

It induces a morphism $\phi : \widetilde{\mathcal{G}} \to \mathrm{pr}_1^* \mathcal{G}(1)$. The last morphism of (ii) is the composition of $\mathrm{pr}_1^* \mathcal{G}(1) \twoheadrightarrow i_{X_0 *}\mathcal{G}$ and $i_{X_0 *}\mathcal{G} \twoheadrightarrow i_{X_0 *}\mathcal{H}$. The cokernel of this morphism has support in $X \times \{0\}$. Besides, the action of t on this cokernel is zero. The restriction of ϕ to the fiber $X_0 = X \times \{0\}$ is the morphism $\mathcal{F} \oplus \mathcal{H} \to \mathcal{G}$, thus the sequence $\widetilde{\mathcal{G}} \to \mathrm{pr}_1^* \mathcal{G}(1) \to i_{X_0 *}\mathcal{H} \to 0$ is exact. The kernel of ϕ, as its cokernel, is an $\mathcal{O}_{X_0}$-module. Thus we can find $\mathcal{Z}$ such that $\ker \phi = i_{X_0 *}\mathcal{Z}$. Since X_0 is a hypersurface of $X \times \mathbb{P}^1$, for every coherent sheaf $\mathcal{L}$ on $X \times \mathbb{P}^1$, we have $\mathrm{Tor}_2(\mathcal{L}, i_{X_0}) = 0$. Applying this to $\mathcal{L} = \widetilde{\mathcal{G}}/_{i_{X_0 *}\mathcal{Z}}$ and using Remark 3, we get $\mathrm{Tor}_1(i_{X_0 *}\mathcal{Z}, i_{X_0}) \subseteq \mathrm{Tor}_1(\widetilde{\mathcal{G}}, i_{X_0}) = \{0\}$. But $\mathrm{Tor}_1(i_{X_0 *}\mathcal{Z}, i_{X_0}) \simeq \mathcal{Z} \otimes N^*_{X_0/X \times \mathbb{P}^1} \simeq \mathcal{Z}$, so $\mathcal{Z} = \{0\}$. □

Recall now that $\mathcal{H} = i_{Y *}\mathcal{E}$ where Y is a smooth hypersurface of X and $\mathcal{E}$ is a locally free sheaf on Y. We consider the space $M_{Y/X}$ of the deformation of the normal cone of Y in X (see [11]); $M_{Y/X}$ is the blowup of $X \times \mathbb{P}^1$ along $Y \times \{0\}$. Let $\sigma : M_{Y/X} \to X \times \mathbb{P}^1$

be the canonical morphism. Then $\sigma^* X_0$ is a reduced divisor in $M_{Y/X}$ with two simple irreducible components: $E = \mathbb{P}\big(N_{Y/X} \oplus \mathcal{O}_Y\big)$ and $D = \mathrm{Bl}_Y\, X \simeq X$, which intersect at $\mathbb{P}\big(N_{Y/X}\big) \simeq Y$. The projection of the blowup from E to $Y \times \{0\}$ will be denoted by q, and the canonical isomorphism from D to $X \times \{0\}$ will be denoted by μ.

We now show:

Lemma 6 *The sheaf $\sigma^* \widetilde{\mathcal{G}}$ is locally free modulo torsion on $M_{Y/X}$, and the associated kernel $\mathcal{N}$ is the push-forward of a locally free sheaf on E. More explicitly, if F is the excess conormal bundle of q, $\mathcal{N} = i_{E*}\,(q^* \mathcal{E} \otimes F)$.*

Proof We start from the exact sequence $0 \to \widetilde{\mathcal{G}} \to \mathrm{pr}_1^* \,\mathcal{G}(1) \to i_{X_0*} \mathcal{H} \to 0$. We define the sheaf $\mathcal{Q}$ by the exact sequence $0 \to \mathcal{Q} \to \sigma^* \,\mathrm{pr}_1^* \,\mathcal{G}(1) \to \sigma^* i_{X_0*} \mathcal{H} \to 0$. Since $\sigma^* i_{X_0*} \mathcal{H}$ is the push-forward of a locally free sheaf on E, the sheaf $\mathcal{Q}$ is locally free on $M_{Y/X}$. Then the following sequence: $0 \to \mathrm{Tor}_1\,(i_{X_0*}\mathcal{H}, \sigma) \to \sigma^* \widetilde{\mathcal{G}} \longrightarrow \mathcal{Q} \to 0$ is exact. The first sheaf being a torsion sheaf, $\mathcal{Q}$ is a locally free quotient of $\widetilde{\mathcal{G}}$ with maximal rank. Besides, using the notations given in the following diagram

$$
\begin{array}{ccc}
E & \overset{i_E}{\hookrightarrow} & M_{Y/X} \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \sigma} \\
Y \times \{0\} & \underset{i_{Y\times\{0\}}}{\hookrightarrow} & X \times \mathbb{P}^1
\end{array}
$$

we have $\mathrm{Tor}_1\,(i_{X_0*}\mathcal{H}, \sigma) = i_{E*}\,(q^* \mathcal{E} \otimes F)$ where F is the excess conormal bundle of q (see [5, Sect. 15]. Be aware of the fact that what we note F is F^* in [5]). $\square$

We consider now the exact sequence $0 \to \mathcal{N} \to \sigma^* \widetilde{\mathcal{G}} \to \mathcal{Q} \to 0$ where $\mathcal{Q}$ is locally free on $M_{Y/X}$ and $\mathcal{N} = i_{E*}\,(q^* \mathcal{E} \otimes F) = i_{E*}\mathcal{L}$. We would like to introduce the class $\mathrm{ch}(\sigma^* \widetilde{\mathcal{G}})$, but it is not defined since $M_{Y/X}$ is of dimension $n+1$. However, $\sigma^* \widetilde{\mathcal{G}}$ fits in a short exact sequence where the Chern classes of the two other sheaves can be defined. Remark that we need Lemma 6 to perform this trick. It cannot be done on $X \times \mathbb{P}^1$ since $\widetilde{\mathcal{G}}$ is torsion-free.

Lemma 7 *Let α be the cohomology class on $M_{Y/X}$ defined by*

$$\alpha = \overline{\mathrm{ch}}(\mathcal{Q}) + i_{E*}\left(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\left(N_{E/M_{Y/X}}\right)^{-1}\right).$$

(i) *The class α is the pullback of a cohomology class on $X \times \mathbb{P}^1$.*
(ii) *We have $i_D^* \alpha = \mu^* \,\mathrm{ch}\left(\widetilde{\mathcal{G}}_0\right)$.*

Proof We compute:

$$\begin{aligned}
i_E^*\alpha &= i_E^*\, i_{E*}\left(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\left(N_{E/M_{Y/X}}\right)^{-1}\right) = \overline{\mathrm{ch}}(\mathcal{L})\ \mathrm{td}\left(N_{E/M_{Y/X}}\right)^{-1}\ c_1\left(N_{E/M_{Y/X}}\right)\\
&= \overline{\mathrm{ch}}(\mathcal{L})\left(1-e^{-c_1\left(N_{E/M_{Y/X}}\right)}\right) = \overline{\mathrm{ch}}(\mathcal{L}) - \overline{\mathrm{ch}}\left(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\right)\\
&= \overline{\mathrm{ch}}\left(i_E^*\mathcal{N}\right) - \overline{\mathrm{ch}}\left(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\right) \quad \text{by Axiom B (vi).}
\end{aligned}$$

From the exact sequence $0\to\mathcal{N}\to\sigma^*\widetilde{\mathcal{G}}\to\mathcal{Q}\to 0$, we get the exact sequence of locally free sheaves on E: $0\to i_E^*\mathcal{N}\to i_E^*\ \sigma^*\widetilde{\mathcal{G}}\to i_E^*\mathcal{Q}\to 0$. We obtain

$$\begin{aligned}
i_E^*\alpha &= \overline{\mathrm{ch}}\left(i_E^*\mathcal{Q}\right) + \overline{\mathrm{ch}}\left(i_E^*\mathcal{N}\right) - \overline{\mathrm{ch}}\left(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\right)\\
&= \overline{\mathrm{ch}}\left(i_E^*\sigma^*\widetilde{\mathcal{G}}\right) - \overline{\mathrm{ch}}\left(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\right)\\
&= \overline{\mathrm{ch}}\left(q^*i_Y^*\mathcal{F}\right) + \overline{\mathrm{ch}}\left(q^*i_Y^*\mathcal{H}\right) - \overline{\mathrm{ch}}\left(\mathcal{L}\otimes N^*_{E/M_{Y/X}}\right)\\
&= q^*\overline{\mathrm{ch}}\left(i_Y^*\mathcal{F}\right) + q^*\overline{\mathrm{ch}}(\mathcal{E}) - \overline{\mathrm{ch}}\left(q^*\mathcal{E}\otimes F\otimes N^*_{E/M_{Y/X}}\right).
\end{aligned}$$

The conormal excess bundle F is the line bundle defined by the exact sequence

$$0\longrightarrow F\longrightarrow q^*N^*_{Y/X\times\mathbb{P}^1}\longrightarrow N^*_{E/M_{Y/X}}\longrightarrow 0.$$

Thus, we have $\det\left(q^*N^*_{Y/X\times\mathbb{P}^1}\right) = F\otimes N^*_{E/M_{Y/X}}$. Besides we have $\det\left(q^*N^*_{Y/X\times\mathbb{P}^1}\right) = q^*\det\left(N^*_{Y/X\times\mathbb{P}^1}\right)$, and we get $i_E^*\alpha = q^*\Big[\overline{\mathrm{ch}}\Big(i_Y^*\mathcal{F}\Big)+\overline{\mathrm{ch}}(\mathcal{E})-\overline{\mathrm{ch}}\Big(\mathcal{E}\otimes\det\left(N^*_{Y/X\times\mathbb{P}^1}\right)\Big)\Big]$. This proves (i).

(ii) The divisors E and D meet transversally. Then

$$\begin{aligned}
i_D^*\alpha &= i_D^*\overline{\mathrm{ch}}(\mathcal{Q}) + i_D^*i_{E*}\left(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\left(N_{E/M_{Y/X}}\right)^{-1}\right)\\
&= \overline{\mathrm{ch}}\left(i_D^*\mathcal{Q}\right) + i_{E\cap D/D*}\ i^*_{E\cap D/E}\left(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\left(N_{E/M_{Y/X}}\right)^{-1}\right)\\
&= \overline{\mathrm{ch}}\left(i_D^*\mathcal{Q}\right) + i_{E\cap D/D*}\left(\overline{\mathrm{ch}}\left(i^*_{E\cap D/E}\mathcal{L}\right)\mathrm{td}\left(N_{E\cap D/D}\right)^{-1}\right)
\end{aligned}$$

by Axiom B (iii). We remark now that $i^*_{E\cap D/E}\mathcal{L} = i^*_{E\cap D}\ \mathcal{N}$. Since $\dim D = n$, we obtain $i_D^*\alpha = \overline{\mathrm{ch}}\left(i_D^*\mathcal{Q}\right) + \mathrm{ch}\left(i_{E\cap D/D*}i^*_{E\cap D}\mathcal{N}\right) = \overline{\mathrm{ch}}\left(i_D^*\mathcal{Q}\right) + \mathrm{ch}\left(i_D^*\mathcal{N}\right)$. Taking the pullback on D, we get an exact sequence $0\to i_D^*\mathcal{N}\to i_D^*\ \sigma^*\widetilde{\mathcal{G}}\to i_D^*\mathcal{Q}\to 0$.

Therefore $i_D^* \sigma^* \widetilde{\mathcal{G}}$ is locally free modulo torsion and, μ being an isomorphism, we have $\overline{\mathrm{ch}}\left(i_D^* \mathcal{Q}\right) + \mathrm{ch}(i_D^* \mathcal{N}) = \mathrm{ch}\left(i_D^* \sigma^* \widetilde{\mathcal{G}}\right) = \mathrm{ch}\left(\mu^* \widetilde{\mathcal{G}}_0\right) = \mu^* \mathrm{ch}\left(\widetilde{\mathcal{G}}_0\right)$. □

Proof of Theorem 4 Let α be the form defined in Lemma 7. Using (i) of this lemma and Axiom B (v), we can write $\alpha = \sigma^* \beta$. Thus $i_D^* \alpha = i_D^* \sigma^* \beta = \mu^* i_{X_0}^* \beta$. By (ii) of the same lemma, $i_D^* \alpha = \mu^* \mathrm{ch}\left(\widetilde{\mathcal{G}}_0\right)$ and we get $i_{X_0}^* \beta = \mathrm{ch}\left(\widetilde{\mathcal{G}}_0\right)$. If $t \in \mathbb{P}^1 \backslash \{0\}$, we have clearly $\beta_{|X_t} = \overline{\mathrm{ch}}(\mathcal{G})$. Since $\beta_{|X_t} = \beta_{|X_0}$, we obtain $\overline{\mathrm{ch}}(\mathcal{G}) = \mathrm{ch}\left(\widetilde{\mathcal{G}}_0\right) = \overline{\mathrm{ch}}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$. □

We can now establish the remaining induction properties.

Theorem 5 *The following assertions are valid*:

(i) *Property* (F_n) *holds.*
(ii) *Property* (P_n) *holds.*

Proof (i) We take $y = [\mathcal{F}]$. Let us first suppose that f is a bimeromorphic map. Then there exists a bimeromorphic map $\sigma : \widetilde{X} \to X$ such that $(f \circ \sigma)^* \mathcal{F}$ is locally free modulo torsion. Then by Proposition 10 (i), $\sigma^* \mathrm{ch}\left(f^! [\mathcal{F}]\right) = \mathrm{ch}\left(\sigma^! f^! [\mathcal{F}]\right) = (f \circ \sigma)^* \mathrm{ch}\, \mathcal{F} = \sigma^* \left[f^* \mathrm{ch}(\mathcal{F})\right]$. Suppose now that f is surjective. Then there exist two bimeromorphic maps $\pi_X : \widetilde{X} \to X$, $\pi_Y : \widetilde{Y} \to Y$ and a surjective map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ such that

- The diagram $\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$ is commutative.
- The sheaf $\pi_Y^* \mathcal{F}$ is locally free modulo torsion.

We can write $\pi_Y^! [\mathcal{F}] = [\mathcal{E}] + \widetilde{y}$ in $G(\widetilde{Y})$, where $\widetilde{y}$ is in the image of the natural map $\iota : G_{\mathrm{tors}}(\widetilde{Y}) \to G(\widetilde{Y})$ and $\mathcal{E}$ is locally free. The functoriality property being known for bimeromorphic maps, it holds for π_X and π_Y. The result is now a consequence of Proposition 2 (iii).

In the general case, we consider the diagram used in the proof of Proposition 2 (iii)

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & W & \xrightarrow{i_W} & \widetilde{Y} \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow[f]{} & f(X) & \longrightarrow & Y \end{array}$$

where $\widetilde{f}$ is surjective. Then the functoriality property holds for $\widetilde{f}$ by the argument above and for i_W by Proposition 10 (ii). This finishes the proof.

(ii) We can suppose that $x = [\mathcal{F}]$, $y = [\mathcal{G}]$ and that $\mathcal{F}$ and $\mathcal{G}$ admit locally free quotients $\mathcal{E}_1, \mathcal{E}_2$ of maximal rank. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the associated kernels. We can also suppose that $\mathrm{supp}(\mathcal{T}_1)$ lies in a simple normal crossing divisor. Then $\mathrm{ch}([\mathcal{F}].[\mathcal{G}]) =$

$\overline{\text{ch}}([\mathcal{E}_1].[\mathcal{E}_2]) + \text{ch}([\mathcal{E}_1].[\mathcal{T}_2]) + \text{ch}([\mathcal{E}_2].[\mathcal{T}_1]) + \text{ch}([\mathcal{T}_1].[\mathcal{T}_2]) = \overline{\text{ch}}(\mathcal{E}_1)\overline{\text{ch}}(\mathcal{E}_2) + \overline{\text{ch}}(\mathcal{E}_1)\,\text{ch}(\mathcal{T}_2) + \overline{\text{ch}}(\mathcal{E}_2)\,\text{ch}(\mathcal{T}_1) + \text{ch}([\mathcal{T}_1].[\mathcal{T}_2])$ by Theorem 4 and Proposition 2 (ii). By dévissage, we can suppose that $\mathcal{T}_1$ is a $\mathcal{O}_Z$-module, where Z is a smooth hypersurface of X. We write $[\mathcal{T}_1] = i_{Z*}u$ and $[\mathcal{T}_2] = v$. Then $[\mathcal{T}_1]\,.\,[\mathcal{T}_2] = i_{Z*}\left(u\,.\,i_Z^!v\right)$. So, by (P_{n-1}), Proposition 6 (ii) and the projection formula,

$$\begin{aligned}\text{ch}\left([\mathcal{T}_1].[\mathcal{T}_2]\right) &= i_{Z*}\left(\text{ch}(u\,.\,i_Z^!v)\,\text{td}\left(N_{Z/X}\right)^{-1}\right)\\ &= i_{Z*}\left(\text{ch}(u)i_Z^*\,\text{ch}(v)\;\;\text{td}\left(N_{Z/X}\right)^{-1}\right)\\ &= i_{Z*}\left(\text{ch}(u)\;\;\text{td}\left(N_{Z/X}\right)^{-1}\right)\text{ch}(v)\\ &= \text{ch}\left(i_{Z*}u\right)\text{ch}(v) = \text{ch}\left([\mathcal{T}_1]\right)\text{ch}\left([\mathcal{T}_2]\right).\end{aligned}$$

□

The proof of Theorem 3 is now concluded.

5 The Grothendieck–Riemann–Roch theorem for projective morphisms

5.1 Proof of the Grothendieck–Riemann–Roch theorem

We have already obtained the Grothendieck–Riemann–Roch theorem for the immersion of a smooth divisor. We reduce now by a blowup the case of the immersion of any smooth submanifold to the divisor case. This construction is classical [5].

Theorem 6 *Let Y be a smooth submanifold of X. Then, for all y in $G(Y)$, we have*

$$\text{ch}\left(i_{Y*}y\right) = i_{Y*}\left(\text{ch}(y)\,\text{td}(N_{Y/X})^{-1}\right).$$

Proof We blow up Y along X as shown below, where E is the exceptional divisor.

$$\begin{array}{ccc} E & \xrightarrow{\;i_E\;} & \widetilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[\;i_Y\;]{} & X \end{array}$$

The exact sequence $0 \to F \to q^*N^*_{Y/X} \to N^*_{E/\widetilde{X}} \to 0$ defines the excess conormal bundle F of q. If d is the codimension of Y in X, then $\text{rank}(F) = d - 1$. Recall the following formulae:

(a) $\forall y \in G(Y)$, $p^!\,i_{Y*}y = i_{E*}\left(q^!y\,.\,\lambda_{-1}F\right)$ (see [15] and in the algebraic case [5]).

(b) $\forall \beta \in A(Y)$, $p^* i_{Y*}\beta = i_{E*}(q^*\beta\, c_{d-1}(F^*))$ (this is Axiom B (v)).
(c) If G is a vector bundle of rank r, then $\operatorname{ch}\left(\lambda_{-1}[G]\right) = c_r(G^*)\operatorname{td}(G^*)^{-1}$ (see [5, Lemme 18]).

We obtain

$$
\begin{aligned}
p^* \operatorname{ch}\left(i_{Y!}y\right) &= \operatorname{ch}\left(p^!\, i_{Y!}y\right) = \operatorname{ch}\left(i_{E*}(q^! y\,.\,\lambda_{-1}[F])\right)\\
&= i_{E*}\left(\operatorname{ch}\left(q^! y\,.\,\lambda_{-1}[F]\right)\operatorname{td}\left(N_{E/\widetilde{X}}\right)^{-1}\right)\\
&= i_{E*}\left(q^*\operatorname{ch}(y)\operatorname{ch}\left(\lambda_{-1}[F]\right)\ q^*\operatorname{td}\left(N_{Y/X}\right)^{-1}\operatorname{td}\left(F^*\right)\right)\\
&= i_{E*}\left(q^*\left(\operatorname{ch}(y)\operatorname{td}\left(N_{Y/X}\right)^{-1}\right)c_{d-1}(F^*)\right)\\
&= p^* i_{Y*}\left(\operatorname{ch}(y)\operatorname{td}\left(N_{Y/X}\right)^{-1}\right).
\end{aligned}
$$

Thus $\operatorname{ch}\left(i_{Y*}y\right) = i_{Y*}\left(\operatorname{ch}(y)\operatorname{td}\left(N_{Y/X}\right)^{-1}\right)$. □

Now we can prove a more general Grothendieck–Riemann–Roch theorem:

Theorem 7 *The Grothendieck–Riemann–Roch theorem holds for projective morphisms between smooth complex compact manifolds.*

Proof Let $f: X \to Y$ be a projective morphism. Then we can write f as the composition of an immersion $i: X \to Y \times \mathbb{P}^N$ and the second projection $p: Y \times \mathbb{P}^N \to Y$. By Theorem 6, the Grothendieck–Riemann–Roch theorem is true for i. Now the arguments in [3] show that the canonical map from $G(Y) \otimes_{\mathbb{Z}} G(\mathbb{P}^N)$ to $G(Y \times \mathbb{P}^N)$ is surjective. Therefore, it is enough to prove the Grothendieck–Riemann–Roch theorem for p with elements of the form $y\,.\,w$, where $y \in G(Y)$ and $w \in G(\mathbb{P}^N)$. By the product formula for the Chern character, we are led to the Hirzebruch–Riemann–Roch formula for $\mathbb{P}^N$, which is Axiom B (vii). □

5.2 Compatibility of Chern classes and the Grothendieck–Riemann–Roch theorem

We will show that the Grothendieck–Riemann–Roch theorem for immersions combined with some basic properties can be sufficient to characterize completely a theory of Chern classes. The following compatibility theorem will apply in various situations:

Theorem 8 *Let $X \mapsto A(X)$ be a cohomology theory on smooth complex compact manifolds which satisfies Axioms C in Sect. 2.1. Let* $\operatorname{ch}, \operatorname{ch}' : G(X) \to A(X)$ *be two group morphisms such that*

(i) ch *and* ch′ *are functorial by pullback under holomorphic maps.*
(ii) *For every line bundle L,* $\operatorname{ch}(L) = \operatorname{ch}'(L)$.

(iii) ch *and* ch′ *verify the Grothendieck–Riemann–Roch theorem for smooth immersions.*

Then ch = ch′.

Remark 4 1. The same conclusion holds for cohomology algebras over $\mathbb{Z}$ if we assume the Grothendieck–Riemann–Roch theorem *without denominators*.
2. If X is projective, (i) and (ii) are sufficient to imply the equality of ch and ch′ because of the existence of global locally free resolutions.

Proof We start by proving that for any holomorphic vector bundle E, $\mathrm{ch}(E)$ and $\mathrm{ch}'(E)$ are equal. We argue by induction on the rank of E. Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle of E. Then we have the exact sequence $0 \to \mathcal{O}_E(-1) \to \pi^* E \to F \to 0$ on $\mathbb{P}(E)$, where F is a holomorphic vector bundle on $\mathbb{P}(E)$ whose rank is the rank of E minus one. By induction, $\mathrm{ch}(F) = \mathrm{ch}'(F)$ and by (ii), $\mathrm{ch}(\mathcal{O}_E(-1)) = \mathrm{ch}'(\mathcal{O}_E(-1))$. Thus, $\mathrm{ch}(\pi^* E) = \mathrm{ch}'(\pi^* E)$ and by (i), $\pi^*[\mathrm{ch}(E) - \mathrm{ch}'(E)] = 0$. By Axiom C (iii), $\mathrm{ch}(E) = \mathrm{ch}'(E)$.

We can now prove Theorem 8. The proof proceeds by induction on the dimension of the base manifold X.

Let $\mathcal{F}$ be a coherent sheaf on X. By Proposition 7 there exists a bimeromorphic morphism $\sigma : \widetilde{X} \to X$ which is a finite composition of blowups with smooth centers and a locally free sheaf $\mathcal{E}$ on $\widetilde{X}$ which is a quotient of maximal rank of $\sigma^* \mathcal{F}$. Furthermore, by Hironaka's theorem, we can suppose that the exceptional locus of σ and the kernel of the morphism $\sigma^* \mathcal{F} \to \mathcal{E}$ are both contained in a simple normal crossing divisor D of $\widetilde{X}$. Thus $\sigma^![\mathcal{F}] = \sum_{i=0}^{n} (-1)^i \,[\mathrm{Tor}_i(\mathcal{F}, \sigma)] = [\mathcal{E}] + \sum_{i=1}^{n} (-1)^i \,[\mathrm{Tor}_i(\mathcal{F}, \sigma)]$ and then $\sigma^![\mathcal{F}] \in [\mathcal{E}] + G_D(\widetilde{X})$. Now there is a surjective morphism $\oplus_{i=1}^{N} G_{D_i}(\widetilde{X}) \to G_D(\widetilde{X})$. Moreover, $G(D_i)$ is isomorphic to $G_{D_i}(\widetilde{X})$. Remark that $\mathrm{td}\left(N_{D_i/\widetilde{X}}\right) = \mathrm{td}'\left(N_{D_i/\widetilde{X}}\right)$. By the Grothendieck–Riemann–Roch theorem and the induction hypothesis, ch and ch′ are equal on each $G_{D_i}(\widetilde{X})$. By the first part of the proof, $\mathrm{ch}(\mathcal{E}) = \mathrm{ch}'(\mathcal{E})$. Thus $\mathrm{ch}\left(\sigma^![\mathcal{F}]\right) = \mathrm{ch}'\left(\sigma^![\mathcal{F}]\right)$. By (ii), $\sigma^*\left[\mathrm{ch}(\mathcal{F}) - \mathrm{ch}'(\mathcal{F})\right] = 0$. Since σ^* is injective by Axiom C (i), $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}'(\mathcal{F})$. □

Corollary 1 *Let $\mathcal{F}$ be a coherent analytic sheaf on X. Then*:

(i) *The Chern character* $\mathrm{ch}(\mathcal{F})$ *in rational Deligne cohomology given by Theorem* 1 *is mapped to the topological Chern character of $\mathcal{F}$ by the natural morphism from* $\oplus_i H_D^{2i}(X, \mathbb{Q}(i))$ *to* $\oplus_i H^{2i}(X, \mathbb{Q})$.

(ii) *The image of* $\mathrm{ch}(\mathcal{F})$ *via the natural morphism from* $\oplus_i H_D^{2i}(X, \mathbb{Q}(i))$ *to the Hodge ring* $\oplus_i H^i(X, \Omega_X^i)$ *is the Atiyah Chern character of $\mathcal{F}$.*

Proof It suffices to notice that the Grothendieck–Riemann–Roch theorem for immersions holds for the topological Chern character by [2] and for Atiyah Chern character by [22]. Thus Theorem 8 applies. □

Remark 5 If X is a Kähler complex manifold, the Green Chern classes are the same as the Atiyah Chern classes and the complex topological Chern classes. If X is non Kähler, the Grothendieck–Riemann–Roch theorem does not seem to be known for the Green Chern character, except for a constant morphism (see [26]). If this were true for immersions, it would imply the compatibility of $\mathrm{ch}(\mathcal{F})$ and $\mathrm{ch}(\mathcal{F})^{\mathrm{Gr}}$, via the map from $\oplus_i H_D^{2i}(X, \mathbb{Q}(i))$ to $\oplus_i \mathbb{H}^{2i}(X, \Omega_X^{\bullet \geq i})$. On the other hand, if this compatibility holds, it implies the Grothendieck–Riemann–Roch theorem for immersions for the Green Chern character.

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