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ON A CONJECTURE OF KASHIWARA RELATING CHERN AND EULER CLASSES OF $\mathcal{O}$-MODULES

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Abstract

In this note we prove a conjecture of Kashiwara, which states that the Euler class of a coherent analytic sheaf $\mathcal{F}$ on a complex manifold X is the product of the Chern character of $\mathcal{F}$ with the Todd class of X. As a corollary, we obtain a functorial proof of the Grothendieck–Riemann–Roch theorem in Hodge cohomology for complex manifolds.

1. Introduction

The notation used throughout this article is defined in §2.

Let X be a complex manifold, ω_X be the holomorphic dualizing complex of X, δ_X be the diagonal injection of X in $X \times X$, and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ be the full subcategory of the bounded derived category of analytic sheaves on X consisting of objects with coherent cohomology. In the letter [**7**] that is reproduced in Chapter 5 of [**6**], Kashiwara constructs for every $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ two cohomology classes $\mathrm{hh}_X(\mathcal{F})$ and $\mathrm{thh}_X(\mathcal{F})$ in $\mathrm{H}^0_{\mathrm{supp}(\mathcal{F})}(X, \delta_X^* \delta_{X*} \mathcal{O}_X)$ and $\mathrm{H}^0_{\mathrm{supp}(\mathcal{F})}(X, \delta_X^! \delta_{X!}\, \omega_X)$; they are the Hochschild and co-Hochschild classes of $\mathcal{F}$.

Let us point out that characteristic classes in Hochschild homology are wellknown in homological algebra (see [**8**, §8]). They have been recently intensively studied in various algebraico-geometric contexts. For further details, we refer the reader to [**3**, **2**, **13**] and to the references therein.

If $f\colon X \longrightarrow Y$ is a holomorphic map, the classes hh_X and thh_X satisfy the following dual functoriality properties:

- For every $\mathcal{G}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$, $\mathrm{hh}_X(f^* \mathcal{G}) = f^* \,\mathrm{hh}_Y(\mathcal{G})$.
- For every $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that f is proper on $\mathrm{supp}(\mathcal{F})$,

$$\mathrm{thh}_Y(Rf_!\, \mathcal{F}) = f_!\, \mathrm{thh}_X(\mathcal{F}).$$

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The analytic Hochschild–Kostant–Rosenberg isomorphisms constructed in [**7**] are specific isomorphisms

$$\delta_X^*\delta_{X*}\mathcal{O}_X \simeq \bigoplus_{i\geq 0}\Omega_X^i[i] \qquad \text{and} \qquad \delta_X^!\delta_{X!}\,\omega_X \simeq \bigoplus_{i\geq 0}\Omega_X^i[i]$$

in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. The Hochschild and co-Hochschild classes of an element $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ are mapped via the above HKR isomorphisms to the so-called Chern and Euler classes of $\mathcal{F}$ in $\bigoplus_{i\geq 0}\mathrm{H}^i_{\mathrm{supp}(\mathcal{F})}(X,\Omega_X^i)$, denoted by $\mathrm{ch}(\mathcal{F})$ and $\mathrm{eu}(\mathcal{F})$.

The natural morphism

$$\bigoplus_{i\geq 0}\mathrm{H}^i_{\mathrm{supp}(\mathcal{F})}(X,\Omega_X^i) \longrightarrow \bigoplus_{i\geq 0}\mathrm{H}^i(X,\Omega_X^i)$$

maps $\mathrm{ch}(\mathcal{F})$ to the usual Chern character of $\mathcal{F}$ in Hodge cohomology, which is obtained by taking the trace of the exponential of the Atiyah class of the tangent bundle TX.[1]

The Chern and Euler classes satisfy the same functoriality properties as the Hochschild and co-Hochschild classes—namely, for every holomorphic map f from X to Y we have the following:

– For every $\mathcal{G}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$, $\mathrm{ch}(f^*\mathcal{G}) = f^*\,\mathrm{ch}(\mathcal{G})$,
– For every $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that f is proper on $\mathrm{supp}(\mathcal{F})$,

$$\mathrm{eu}(Rf_!\,\mathcal{F}) = f_!\,\mathrm{eu}(\mathcal{F}).$$

Furthermore, for every $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, $\mathrm{eu}(\mathcal{F}) = \mathrm{ch}(\mathcal{F})\,\mathrm{eu}(\mathcal{O}_X)$. Putting together the previous identity with the functoriality of the Euler class with respect to direct images, Kashiwara obtained the following Grothendieck–Riemann–Roch theorem:

Theorem 1.1. [**7**] *Let $f : X \longrightarrow Y$ be a holomorphic map and $\mathcal{F}$ be an element of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that f is proper on $\mathrm{supp}(\mathcal{F})$. Then the following identity holds in $\bigoplus_{i\geq 0}\mathrm{H}^i_{f[\mathrm{supp}(\mathcal{F})]}(Y,\Omega_Y^i)$:*

$$\mathrm{ch}(Rf_!\,\mathcal{F})\,\mathrm{eu}(\mathcal{O}_Y) = f_!\left[\mathrm{ch}(\mathcal{F})\,\mathrm{eu}(\mathcal{O}_X)\right].$$

Then Kashiwara stated the following conjecture (see [**6**, §5.3.4]):

Conjecture 1.2. [**7**] For any complex manifold X, the class $eu(\mathcal{O}_X)$ is the Todd class of the tangent bundle TX.

This conjecture was related to another conjecture of Schapira and Schneiders comparing the Euler class of a $\mathscr{D}_X$-module $\mathfrak{m}$ and the Chern class of the associated $\mathcal{O}_X$-module $\mathrm{Gr}(\mathfrak{m})$ (see [**12**, **1**]).

[1] This property has been proved in [**2**] for algebraic varieties using different definitions of the HKR isomorphism and of the Hochschild class. In Kashiwara's setting, this is straightforward.

The aim of this note is to give a simple proof of Kashiwara's conjecture:

Theorem 1.3. *For any complex manifold X,* $\mathrm{eu}(\mathcal{O}_X)$ *is the Todd class of TX.*

In the algebraic setting, an analogous result is established in [**11**] (see also [**9**]).

As a corollary of Theorem 1.3, we obtain the Grothendieck-Riemann-Roch theorem in Hodge cohomology for abstract complex manifolds, which has been already proved by different methods in [**10**]:

Theorem 1.4. *Let $f\colon X \longrightarrow Y$ be a holomorphic map between complex manifolds, and let $\mathcal{F}$ be an element of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ such that f is proper on* $\mathrm{supp}(\mathcal{F})$*. Then*

$$\mathrm{ch}(Rf_!\,\mathcal{F})\,\mathrm{td}(Y) = f_!\left[\mathrm{ch}(\mathcal{F})\,\mathrm{td}(X)\right]$$

in $\bigoplus_{i\geq 0} \mathrm{H}^i_{f[\mathrm{supp}(\mathcal{F})]}(Y, \Omega^i_Y)$.

However, the proof given here is simpler and more conceptual. Besides, we would like to emphasize that it is entirely self-contained and relies only on the results appearing in Chapter 5 of [**6**].

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2. Notations and basic results

We follow the notation of [**6**, Ch.5].

If X is a complex manifold, we denote by $\mathrm{D}^{\mathrm{b}}(X)$ the bounded derived category of sheaves of $\mathcal{O}_X$-modules and by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ consisting of complexes with coherent cohomology.

If $f\colon X \longrightarrow Y$ is a holomorphic map between complex manifolds, the four operations $f^*\colon \mathrm{D}^{\mathrm{b}}(Y) \longrightarrow \mathrm{D}^{\mathrm{b}}(X)$, Rf_*, $Rf_!\colon \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(Y)$, and $f^!\colon \mathrm{D}^{\mathrm{b}}(Y) \longrightarrow \mathrm{D}^{\mathrm{b}}(X)$ are part of the formalism of Grothendieck's six operations. Let us recall their definitions:

- f^* is the left derived functor of the pullback functor by f, that is, $\mathcal{G} \to \mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.
- Rf_* is the right derived functor of the direct image functor f_*, it is the left adjoint to the functor f^*.
- $Rf_!$ is the right derived functor of the proper direct image functor $f_!$.
- $f^!$ is the exceptional inverse image; it is the right adjoint to the functor $Rf_!$.

If W is a closed complex submanifold of Y, the pullback morphism from $f^*\Omega_Y^i[i]$ to $\Omega_X^i[i]$ induces in cohomology a map

$$f^*\colon \bigoplus_{i\geq 0} \mathrm{H}^i_W(Y,\Omega_Y^i) \longrightarrow \bigoplus_{i\geq 0} \mathrm{H}^i_{f^{-1}(W)}(X,\Omega_X^i).$$

If Z is a closed complex submanifold of X and if f is proper on Z, the integration morphism from $\Omega_X^{i+d_X}[i+d_X]$ to $\Omega_Y^{i+d_Y}[i+d_Y]$ induces a Gysin morphism

$$f_!:\bigoplus_{i\geq -d_X} \mathrm{H}^{i+d_X}_Z(X,\Omega_X^{i+d_X}) \longrightarrow \bigoplus_{i\geq -d_Y} \mathrm{H}^{i+d_Y}_{f(Z)}(Y,\Omega_Y^{i+d_Y}).$$

Let X be a complex manifold, ω_X be the holomorphic dualizing complex of X, and δ_X be the diagonal injection. If $\mathcal{F}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, we define the ordinary dual (resp. Verdier dual) of $\mathcal{F}$ by the usual formula $D'\mathcal{F} = \mathcal{RH}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{O}_X)$ (resp. $D\mathcal{F} = \mathcal{RH}om_{\mathcal{O}_X}(\mathcal{F},\omega_X)$).

The Hochschild and co-Hochschild classes of $\mathcal{F}$, denoted by $\mathrm{hh}_X(\mathcal{F})$ and $\mathrm{thh}_X(\mathcal{F})$, lie in $\mathrm{H}^0_{\mathrm{supp}(\mathcal{F})}(X,\delta_X^*\delta_{X*}\mathcal{O}_X)$ and $\mathrm{H}^0_{\mathrm{supp}(\mathcal{F})}(X,\delta_X^!\delta_{X!}\,\omega_X)$, respectively. They are constructed by the chains of maps

$$\begin{aligned}\mathrm{hh}_X(\mathcal{F}):\ \ \mathrm{id} &\longrightarrow \mathcal{RH}om(\mathcal{F},\mathcal{F}) \xrightarrow{\sim} \delta_X^*(D'\mathcal{F}\boxtimes_{\mathcal{O}}\mathcal{F}) \longrightarrow \delta_X^*\delta_{X*}\,\mathcal{O}_X,\\ \mathrm{thh}_X(\mathcal{F}):\ \ \mathrm{id} &\longrightarrow \mathcal{RH}om(\mathcal{F},\mathcal{F}) \xrightarrow{\sim} \delta_X^!(D\mathcal{F}\boxtimes_{\mathcal{O}}\mathcal{F}) \longrightarrow \delta_X^!\delta_{X!}\,\omega_X\end{aligned}$$

where in both cases the last arrows are obtained from the derived trace maps

$$D'\mathcal{F}\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{F}\longrightarrow\mathcal{O}_X \quad \text{and} \quad D\mathcal{F}\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\mathcal{F}\longrightarrow\omega_X$$

by adjunction.

If $f: X\longrightarrow Y$ is a holomorphic map between complex manifolds, there are pullback and push-forward morphisms

$$f^*\colon f^*\delta_Y^*\delta_{Y*}\,\mathcal{O}_Y\longrightarrow\delta_X^*\delta_{X*}\,\mathcal{O}_X \quad\text{and}\quad f_!: Rf_!\,\delta_X^!\delta_{X!}\omega_X\longrightarrow\delta_Y^!\delta_{Y!}\omega_Y.$$

Besides, there is a natural pairing

$$\delta_X^*\delta_{X*}\,\mathcal{O}_X\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\delta_X^!\delta_{X!}\,\omega_X \ \to\ \delta_X^!\delta_{X!}\,\omega_X \tag{1}$$

given by the chain

$$\delta_X^*\delta_{X*}\,\mathcal{O}_X\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}\delta_X^!\delta_{X!}\,\omega_X \simeq \delta_X^!(\delta_{X*}\,\mathcal{O}_X\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{X\times X}}\delta_{X!}\,\omega_X)\ \to\ \delta_X^!\delta_{X!}\,\omega_X.$$

Theorem 2.1. [**7**]

(i) *For all elements $\mathcal{F}$ and $\mathcal{G}$ in $D^{\mathrm{b}}_{\mathrm{coh}}(X)$ and $D^{\mathrm{b}}_{\mathrm{coh}}(Y)$ respectively,* $\mathrm{hh}_X(f^*\mathcal{G}) = f^*\,\mathrm{hh}_Y(\mathcal{G})$ *and* $f_!\,\mathrm{thh}_X(\mathcal{F}) = \mathrm{thh}_Y(Rf_!\mathcal{F})$.

(ii) *Via the pairing* (1), *for every* $\mathcal{F}$ *in* $D^{\mathrm{b}}_{\mathrm{coh}}(X)$,

$$\mathrm{hh}_X(\mathcal{F})\,\mathrm{thh}(\mathcal{O}_X) = \mathrm{thh}_X(\mathcal{F}).$$

The Hochschild and co-Hochschild classes are translated into Hodge cohomology classes by Kashiwara's analytic Hochschild–Kostant–Rosenberg isomorphisms[2]

$$(2) \qquad \delta_X^* \delta_{X*} \mathcal{O}_X \simeq \bigoplus_{i\geq 0} \Omega_X^i[i] \qquad \text{and} \qquad \delta_X^! \delta_{X!}\, \omega_X \simeq \bigoplus_{i\geq 0} \Omega_X^i[i],$$

and the resulting classes are called Chern and Euler classes.If $\mathcal{F}$ is an element of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, then $\mathrm{ch}(\mathcal{F})$ and $\mathrm{eu}(\mathcal{F})$ lie in $\bigoplus_{i\geq 0} \mathrm{H}^i_{\mathrm{supp}(\mathcal{F})}(X, \Omega_X^i)$.

The first HKR isomorphism commutes with pullback and the second one with push forward. Besides, the pairing (1) between $\delta_X^* \delta_{X*}\, \mathcal{O}_X$ and $\delta_X^! \delta_{X!}\, \omega_X$ is exactly the cup-product on holomorphic differential forms after applying the HKR isomorphisms (2).

Theorem 2.2. [**7**]

(i) *For every* $\mathcal{F}$ *in* $D^{\mathrm{b}}_{\mathrm{coh}}(X)$, $\mathrm{ch}(\mathcal{F})$ *is the usual Chern character obtained by the Atiyah exact sequence.*
(ii) *For all elements* $\mathcal{F}$ *and* $\mathcal{G}$ *in* $D^{\mathrm{b}}_{\mathrm{coh}}(X)$ *and* $D^{\mathrm{b}}_{\mathrm{coh}}(X)$ *respectively,* $\mathrm{ch}(f^*\mathcal{G}) = f^* \mathrm{ch}(\mathcal{G})$ *and* $f_! \,\mathrm{eu}(\mathcal{F}) = \mathrm{eu}(Rf_! \mathcal{F})$.
(iii) *For every* $\mathcal{F}$ *in* $D^{\mathrm{b}}_{\mathrm{coh}}(X)$, $\mathrm{eu}(\mathcal{F}) = \mathrm{ch}(\mathcal{F})\,\mathrm{eu}(\mathcal{O}_X)$.

For the proofs of Theorems 2.1 and 2.2, we refer to [**6**, Ch. 5].

For any complex manifold X, we denote by $\mathrm{td}(X)$ the Todd class of the tangent bundle TX in $\bigoplus_{i\geq 0} \mathrm{H}^i(X, \Omega_X^i)$.

3. Proof of Theorem 1.3

We proceed in several steps.

Proposition 3.1. *Let* Y *and* Z *be complex manifolds such that* Z *is a closed complex submanifold of* Y, *and let* i_Z *be the corresponding inclusion. Then, for every coherent sheaf* $\mathcal{F}$ *on* Z, *we have*

$$i_{Z!}\,[\mathrm{ch}(\mathcal{F})\,\mathrm{td}(Z)] = \mathrm{ch}(i_{Z*}\mathcal{F})\,\mathrm{td}(Y)$$

in $\bigoplus_{i\geq 0} \mathrm{H}^i_Z(Y, \Omega_Y^i)$.

Proof. This is proved in the classical way using the deformation to the normal cone as in [**4**, §15.2], except that we use local cohomology. For the sake of completeness, we provide a detailed proof.

We start by a particular case:

– $\mathcal{N}$ is a holomorphic vector bundle on Z, and $Y = \mathbb{P}\,(\mathcal{N} \oplus \mathcal{O}_Z)$.

[2]For a detailed account of the HKR isomorphisms, we refer to the introduction of [**5**] and to the references therein.

– Z embeds in Y by identifying Z with the zero section of $\mathcal{N}$.

Let d be the rank of $\mathcal{N}$, π be the projection of the projective bundle $\mathbb{P}(\mathcal{N}\oplus\mathcal{O}_Z)$, and $\mathcal{Q}$ be the universal quotient bundle on Y; $\mathcal{Q}$ is the quotient of $\pi^*(\mathcal{N}\oplus\mathcal{O}_Z)$ by the tautological line bundle $\mathcal{O}_{\mathcal{N}\oplus\mathcal{O}_Z}(-1)$. Then $\mathcal{Q}$ has a canonical holomorphic section s that is obtained by the composition

$$s\colon \mathcal{O}_Y \simeq \pi^*\mathcal{O}_Z \longrightarrow \pi^*(\mathcal{N}\oplus\mathcal{O}_Z) \longrightarrow \mathcal{Q}.$$

This section vanishes transversally along its zero locus, which is exactly Z. Therefore, we have a natural locally free resolution of $i_{Z!}\mathcal{O}_Z$ given by the Koszul complex associated with the pair $(\mathcal{Q}^*, s^*)$:

$$0 \longrightarrow \wedge^d\mathcal{Q}^* \longrightarrow \wedge^{d-1}\mathcal{Q}^* \longrightarrow \cdots \longrightarrow \mathcal{O}_Y \longrightarrow i_{Z!}\mathcal{O}_Z \longrightarrow 0.$$

This gives the equality

$$\mathrm{ch}(i_{Z!}\mathcal{O}_Z) = \sum_{k=0}^{d}(-1)^k\mathrm{ch}(\wedge^k\mathcal{Q}^*) = \mathrm{c}_d(\mathcal{Q})\,\mathrm{td}(\mathcal{Q})^{-1}$$

in $\bigoplus_{i\geq 0}\mathrm{H}^i(Y,\Omega^i_Y)$, where $\mathrm{c}_d(\mathcal{Q})$ denotes the dth Chern class of $\mathcal{Q}$ (for the last equality, see [**4**, § 3.2.5]). Since $\mathrm{c}_d(\mathcal{Q})$ is the image of the constant class 1 by $i_{Z!}$ and since $i_Z^*\mathcal{Q}=\mathcal{N}$, we get

$$\mathrm{ch}(i_{Z!}\mathcal{O}_Z) = i_{Z!}(i_Z^*\mathrm{td}(\mathcal{Q})^{-1}) = i_{Z!}(\mathrm{td}(\mathcal{N})^{-1}).$$

For any coherent sheaf $\mathcal{F}$ on Z, we have $i_{Z!}\mathcal{F} = i_{Z!}\mathcal{O}_Z \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y}\pi^*\mathcal{F}$ so that we obtain by the projection formula

$$\mathrm{ch}(i_{Z!}\mathcal{F}) = i_{Z!}(\mathrm{ch}(\mathcal{F})\mathrm{td}(\mathcal{N})^{-1}) \tag{3}$$

in $\bigoplus_{i\geq 0}\mathrm{H}^i(Y,\Omega^i_Y)$. Remark now that by Theorem 2.2 (ii) and (iii), we have

$$\mathrm{ch}(i_{Z!}\,\mathcal{F}) = i_{Z!}(\mathrm{ch}(\mathcal{F})\mathrm{eu}(\mathcal{O}_Z)\,i_Z^*\,\mathrm{eu}(\mathcal{O}_Y)^{-1})$$

in $\bigoplus_{i\geq 0}\mathrm{H}^i_Z(Y,\Omega^i_Y)$. This proves that $\mathrm{ch}(i_{Z!}\,\mathcal{F})$ lies in the image of

$$i_{Z!}\colon \bigoplus_{i\geq 0}\mathrm{H}^i(Z,\Omega^i_Z) \longrightarrow \bigoplus_{i\geq 0}\mathrm{H}^{i+d}_Z(Y,\Omega^{i+d}_Y).$$

Let us denote this image by W. The map

$$\iota\colon W \longrightarrow \bigoplus_{i\geq 0}\mathrm{H}^{i+d}(Y,\Omega^{i+d}_Y)$$

obtained by forgetting the support is injective. Indeed, for every class $i_{Z!}\alpha$ in W, $\pi_![\iota(i_{Z!}\alpha)]=\alpha$. This implies that (3) holds in $\bigoplus_{i\geq 0}\mathrm{H}^i_Z(Y,\Omega^i_Y)$.

We now turn to the general case, using deformation to the normal cone. Let us introduce some notation:

- M is the blowup of $Z\times\{0\}$ in $Y\times\mathbb{P}^1$,$and\sigma$ is the blowup map and $q=\mathrm{pr}_1\circ\sigma$.
- For any divisor D on M, $[D]$ denotes its cohomological cycle class in $\mathrm{H}^1(M,\Omega^1_M)$.
- $E=\mathbb{P}(N_{Z/Y}\oplus\mathcal{O}_Z)$ is the exceptional divisor of the blowup, and $\widetilde{Y}$ is the strict transform of $Y\times\{0\}$ in M.
- μ is the embedding of Z in E, where Z is identified with the zero section of $N_{Z/Y}$.
- F is the embedding of (the strict transform of) $Z\times\mathbb{P}^1$ in M, and, for any t in $\mathbb{P}^1$, j_t is the embedding of M_t in M.
- k is the embedding of E in M.

Then M is flat over $\mathbb{P}^1$, M_0 is a Cartier divisor with two smooth components E and $\widetilde{Y}$ intersecting transversally along $\mathbb{P}(N_{Z/Y})$, and M_t is isomorphic to Y if t is nonzero.

Let $\mathcal{G}=F_!(\mathrm{pr}_1^*\mathcal{F})$. Since M is flat over $\mathbb{P}^1$, for any t in $\mathbb{P}^1\setminus\{0\}$,

$$j_t^*\mathcal{G}=i_{Z!}\mathcal{F}\quad\text{and}\quad k^*\mathcal{G}=\mu_!\mathcal{F}.$$

If $\mathrm{ch}(\mathcal{G})$ is the Chern character of $\mathcal{G}$ in $\bigoplus_{i\geq 0}\mathrm{H}^i_{Z\times\mathbb{P}^1}(M,\Omega^i_M)$, using the identity (3) in $\bigoplus_{i\geq 0}\mathrm{H}^i_Z(E,\Omega^i_E)$, we get

$$\begin{aligned}
j_{t!}\,\mathrm{ch}(i_{Z!}\mathcal{F}) &= j_{t!}\,j_t^*\,\mathrm{ch}(\mathcal{G})=\mathrm{ch}(\mathcal{G})\,[M_t]\\
&=\mathrm{ch}(\mathcal{G})\,[M_0]=\mathrm{ch}(\mathcal{G})\,[E]+\mathrm{ch}(\mathcal{G})\,[\widetilde{Y}]\\
&=\mathrm{ch}(\mathcal{G})\,[E]=k_!\,k^*\,\mathrm{ch}(\mathcal{G})=k_!\,\mathrm{ch}(\mu_!\mathcal{F})\\
&=k_!\,\mu_!(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/E})^{-1})\\
&=k_!\,\mu_!(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1})
\end{aligned}$$

in $\bigoplus_{i\geq 0}\mathrm{H}^i_{Z\times\mathbb{P}^1}(M,\Omega^i_M)$.

The map q is proper on $Z\times\mathbb{P}^1$, $q\circ j_t=\mathrm{id}$ and $q\circ k\circ\mu=i_Z$. Applying $q_!$, we get

$$\mathrm{ch}(i_{Z!}\mathcal{F})=i_{Z!}(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1})$$

in $\bigoplus_{i\geq 0}\mathrm{H}^i_Z(Y,\Omega^i_Y)$. q.e.d.

Definition 3.2. For any complex manifold X, let $\alpha(X)$ be the cohomology class in $\bigoplus_{i\geq 0}\mathrm{H}^i(X,\Omega^i_X)$ defined by $\alpha(X)=\mathrm{eu}(\mathcal{O}_X)\,\mathrm{td}(X)^{-1}$.

Lemma 3.3. *Let Y and Z be complex manifolds such that Z is a closed complex submanifold of Y, and let i_Z be the corresponding injection. Assume that there exists a holomorphic retraction R of i_Z. Then we have $\alpha(Z)=i_Z^*\,\alpha(Y)$.*

Proof. By Theorem 2.2 (ii), $\mathrm{eu}(i_{Z*}\mathcal{O}_Z) = i_{Z!}\,\mathrm{eu}(\mathcal{O}_Z)$. By Proposition 3.1 and Theorem 2.2 (iii),

$$\mathrm{eu}(i_{Z*}\mathcal{O}_Z) = \mathrm{ch}(i_{Z*}\mathcal{O}_Z)\mathrm{eu}(\mathcal{O}_Y) = (i_{Z!}\,\mathrm{td}(Z))\,\mathrm{td}(Y)^{-1}\mathrm{eu}(\mathcal{O}_Y),$$

so that we obtain in $\bigoplus_{i\geq 0} \mathrm{H}^i_Z(Y, \Omega^i_Y)$ the formula

$$i_{Z!}\big[\mathrm{eu}(\mathcal{O}_Z) - \mathrm{td}(Z)\, i_Z^*(\mathrm{eu}(\mathcal{O}_Y)\,\mathrm{td}(Y)^{-1})\big] = 0.$$

Since R is proper on Z, we can apply $R_!$ and we get the result. q.e.d.

Lemma 3.4. *The class* $\alpha(X)$ *satisfies* $\alpha(X)^2 = \alpha(X)$.

Proof. We apply Lemma 3.3 with $Z = X$ and $Y = X \times X$, where X is diagonally embedded in $X \times X$. Then $\alpha(X) = i^*_\Delta\, \alpha(X \times X)$. The Euler class commutes with external products so that

$$\mathrm{eu}(\mathcal{O}_{X\times X}) = \mathrm{eu}(\mathcal{O}_X) \boxtimes \mathrm{eu}(\mathcal{O}_X).$$

Thus, $\alpha(X \times X) = \alpha(X) \boxtimes \alpha(X)$ and we obtain

$$\alpha(X) = i^*_\Delta[\,\alpha(X) \boxtimes \alpha(X)\,] = \alpha(X)^2.$$

q.e.d.

Proof of Theorem 1.3. There is a natural isomorphism ϕ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ between $\delta^*\delta_*\mathcal{O}_X$ and $\delta^!\delta_!\omega_X$ given by the chain

$$\delta^*\delta_*\mathcal{O}_X \simeq \mathcal{O}_X \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \delta^*\delta_*\mathcal{O}_X \simeq \delta^!(\omega_X \boxtimes \mathcal{O}_X) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \delta^*\delta_*\mathcal{O}_X \simeq \delta^!\delta_!\omega_X.$$

Besides, after applying the two HKR isomorphisms (2), ϕ is given by derived cup-product with the Euler class of $\mathcal{O}_X$ (see [**6**]). Therefore, the class $\mathrm{eu}(\mathcal{O}_X)$ is invertible in the Hodge cohomology ring of X, and so is $\alpha(X)$. Lemma 3.4 implies that $\alpha(X) = 1$. q.e.d.

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