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Topological properties of Hilbert schemes of almost-complex fourfolds (I)

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Abstract. In this article, we study topological properties of Voisin's Hilbert schemes of an almost-complex four-manifold X. We compute in this setting their Betti numbers and construct Nakajima operators. We also define tautological bundles associated with any complex bundle on X, which are shown to be canonical in K–theory.

1. Introduction

The aim of the present article is to extend some constructions and results related to Hilbert schemes of points for projective surfaces to the case of almost-complex compact manifolds of real dimension four.

If X is any smooth irreducible complex projective surface and if n is any positive integer, the Hilbert scheme of points $X^{[n]}$ is the set of all zero-dimensional subschemes of X of length n. By a result of Fogarty [8], $X^{[n]}$ is a smooth irreducible projective variety of complex dimension $2n$. This implies that $X^{[n]}$ can be seen as a smooth compactification of the set of distinct unordered n–tuples of points in X. Besides, if $X^{(n)}$ denotes the n–fold symmetric power of X, the Hilbert–Chow map $\Gamma\colon\ X^{[n]} \longrightarrow X^{(n)}$ defined by $\Gamma(\xi) = \sum_{x\in \operatorname{supp}(\xi)} l_x(\xi)x$ is a resolution of singularities of $X^{(n)}$.

In the papers [20] and [21], Voisin constructs Hilbert schemes $X^{[n]}$ associated with any almost-complex compact four-manifold. Each Hilbert scheme $X^{[n]}$ is a compact differentiable manifold of dimension $4n$ endowed with a stable almost-complex structure, and there exists a continuous Hilbert–Chow map Γ from $X^{[n]}$ to $X^{(n)}$ whose fibers are homeomorphic to the fibers of the usual Hilbert–Chow map. Furthermore, if X is a symplectic compact four-manifold, the differentiable Hilbert schemes $X^{[n]}$ are also symplectic (this is a differentiable analog of a result of Beauville [1]).

Using ideas of Voisin concerning relative integrable complex structures, which are the main technical ingredient of [20], we study the local topological structure of the Hilbert–Chow map. This allows us to compute the Betti numbers of $X^{[n]}$, which extends Göttsche's classical formula [10].

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Theorem 1. *Let (X, J) be an almost-complex compact four-manifold and, for any positive integer n, let $\left(b_i(X^{[n]})\right)_{i=0,\dots,4n}$ be the sequence of Betti numbers of the almost-complex Hilbert scheme $X^{[n]}$. Then the generating function for these Betti numbers is given by the formula*

$$\sum_{n=0}^{\infty}\sum_{i=0}^{4n} b_i\,(X^{[n]})\,t^i q^n = \prod_{m=1}^{\infty} \frac{\left[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\right]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

The proof of Theorem 1 follows closely the argument of Göttsche and Soergel [11] and relies on the decomposition theorem of Deligne et al. [2], which describes derived direct images of DGM-sheaves under proper morphisms between complex algebraic varieties (see [5, Th. 5.4.10]). If $f\colon Y \longrightarrow Z$ is a proper semi-small morphism and if Y is rationally smooth, the decomposition theorem has a particularly nice form: it gives a canonical quasi-isomorphism between $Rf_*\mathbb{Q}_Y$ and a direct sum of explicit intersection complexes on Z. This statement has been proved by Le Potier [15] using neither characteristic p methods nor étale cohomology (which are heavily used in [2]), and his proof can be naturally extended to the case of continuous maps which are locally equivalent on the base to semi-small maps. Since Le Potier's proof is enlightening in many respects and appears only in unpublished lecture notes, we take the opportunity to reproduce it here in an appendix.

The second part of the article is devoted to the definition and the study of the Nakajima operators associated with an arbitrary almost-complex compact four-manifold X. These operators act on the total cohomology ring $\bigoplus_{n\in\mathbb{N}} H^*(X^{[n]}, \mathbb{Q})$ of the Hilbert schemes of points by correspondences associated with incidence varieties. We prove that the Nakajima commutation relations established in [17] can be extended to the almost-complex setting:

Theorem 2. *For any pair (i, j) of integers and any pair (α, β) of cohomology classes in $H^*(X, \mathbb{Q})$ we have*

$$\left[\mathfrak{q}_i(\alpha), \mathfrak{q}_j(\beta)\right] = i\,\delta_{i+j,0}\left(\int_X \alpha\beta\right)\mathrm{id}.$$

It follows from Theorems 1 and 2 that the Nakajima operators define a highest weight irreducible representation of the Heisenberg super-algebra $\mathcal{H}(H^*(X, \mathbb{Q}))$ on the infinite-dimensional graded vector space $\bigoplus_{n\in\mathbb{N}} H^*(X^{[n]}, \mathbb{Q})$.

In the last part of the article, we carry out the construction of tautological bundles $E^{[n]}$ on the complex Hilbert schemes $X^{[n]}$ associated with any complex vector bundle E on an almost-complex compact four-manifold X. To do so, we use relative holomorphic structures on E in the same spirit as the relative integrable complex structures considered in [20].

Let us now give an idea of our strategy to prove the results. If (X, J) is an almost-complex compact manifold and n is a positive integer, Voisin's construction of the Hilbert scheme of n–points associated with X is not canonical and depends on the choice of a relative integrable complex structure $J^{\,\mathrm{rel}}$ on X parameterized by the n–th symmetric power $X^{(n)}$ of X (which means essentially that for all $\underline{x}$ in $X^{(n)}$, $J^{\,\mathrm{rel}}_{\underline{x}}$ is an integrable complex structure in a neighbourhood of the points of $\underline{x}$ in X

varying smoothly with $\underline{x}$). If $J^{\,\mathrm{rel}}$ is such a relative structure and if $X^{[n]}_{J^{\mathrm{rel}}}$ is the associated Hilbert scheme, then $X^{[n]}_{J^{\,\mathrm{rel}}}$ is naturally a differentiable manifold provided that $J^{\,\mathrm{rel}}$ satisfies some additional complicated geometric conditions. We don't use at all the differentiable structure of $X^{[n]}_{J^{\,\mathrm{rel}}}$ in the article, because it is not necessary for our purpose: indeed, the cohomology rings of the Hilbert schemes only depend on their homotopy type.

In Sect. 3 we study, first locally and then globally, relative integrable structures. The local study is achieved in Sect. 3.1 using the existence of relative holomorphic coordinates for relative integrable complex structures. It allows us to prove that $X^{[n]}_{J^{\,\mathrm{rel}}}$ is locally homeomorphic to the integral model $(\mathbb{C}^2)^{[n]}$, and is therefore a topological manifold. As explained earlier, Göttsche's formula (proved in Sect. 3.2) can be deduced thereof. The aim of the global study of relative integrable complex structures on X (Sect. 3.3) is to show that the homeomorphism type of $X^{[n]}_{J^{\,\mathrm{rel}}}$ is independent of $J^{\,\mathrm{rel}}$. Using standard gluing techniques, we prove a slightly stronger result, namely that the Hilbert schemes $X^{[n]}_{J^{\,\mathrm{rel}}}$ vary topologically trivially if $J^{\,\mathrm{rel}}$ varies smoothly. In the case of relative integrable complex structures considered in [20], this fact is a straightforward consequence of Ehresmann's fibration theorem; but this argument is no longer valid for arbitrary relative integrable complex structures.

Our main concern in Sect. 4 is to define and study incidence varieties. Even if a variant of Voisin's relative construction can be used to define these objects for almost-complex four-manifolds (this is done in Sect. 4.1), some new difficulties appear. Indeed the incidence variety $X^{[n',n]}$ cannot be naturally embedded into a product of Hilbert schemes, even for well-chosen relative complex structures. This problem is solved using the product Hilbert schemes $X^{[n]\times[n']}$ introduced at the end of Sect. 3.3. As it is the case for Hilbert schemes, the incidence variety $X^{[n',n]}$ is locally homeomorphic to the integrable model $(\mathbb{C}^2)^{[n',n]}$, which allows to endow $X^{[n',n]}$ with a topological stratification locally modeled on the standard stratification of an analytic set. Thus, incidence varieties carry a fundamental homology class and Nakajima operators can be defined via the action by correspondence of incidence varieties. In Sect. 4.2, we prove that these operators satisfy the Heisenberg commutation relations. Our proof follows closely Nakajima's original one, even if compatibility problems between various relative integrable complex structures make our argument somehow heavy. The main idea underlying the proof is that intersection of cycles can be understood locally as soon as there is no excess intersection components (such a component would yield an excess cohomology class on the set-theoretical intersection of the cycles, which is no longer a local datum). There is only one case where excess contributions appear, corresponding to the commutator $[\mathfrak{q}_i(\alpha), \mathfrak{q}_{-i}(\beta)]$. However, this case can be handled easily because the excess term becomes simply a multiplicity at the end of the computation, hence is a local datum.

In Sect. 5.1, we develop the theory of relative holomorphic structures (or equivalently of relative holomorphic connections) on any complex vector bundle E on X. This allows us to define a tautological vector bundle $E^{[n]}$ on $X^{[n]}$ (which is a new construction, even if J is integrable, because E is not assumed to be holomorphic). We prove that the class of $E^{[n]}$ in $K(X^{[n]})$ is independent of the auxiliary structures used to define it, and we compute in geometric terms the first Chern class of the vector bundle $\mathbb{T}^{[n]}$, where $\mathbb{T}$ is the trivial complex line bundle on X.

In Sect. 5.2, using methods already present in Sect. 4, we establish the induction relation comparing the classes of $E^{[n]}$ and $E^{[n+1]}$ through the incidence variety $X^{[n+1,\,n]}$.

2. Notations and conventions

Throughout the article, (X, J) is an almost-complex compact four-manifold.

2.1. Symmetric powers

- If n is a positive integer, let $\mathfrak{S}_n$ be the symmetric group on n symbols. The *n–fold symmetric power* $X^{(n)}$ of X is the quotient of X^n by the action of $\mathfrak{S}_n$. It is endowed with the sheaf $\mathcal{C}^{\infty}_{X^{(n)}}$ of smooth functions on X^n invariant under $\mathfrak{S}_n$.
- Elements of the symmetric powers appear most of the time as underlined letters. For any $\underline{x}$ in $X^{(n)}$, the *support* of $\underline{x}$ is the subset of X consisting of points in $\underline{x}$; we denote it by $\mathrm{supp}(\underline{x})$.
- For any positive integers p and q, we denote by $(\underline{x}, \underline{y}) \longmapsto \underline{x} \cup \underline{y}$ the natural map from $X^{(p)} \times X^{(q)}$ to $X^{(p+q)}$.
- For any positive integer n and any x in X, we denote by nx the unique element in $X^{(n)}$ satisfying $\mathrm{supp}(nx) = \{x\}$.
- If $k, n_1, \ldots, n_k$ are positive integers, the *incidence set* $Z_{n_1\times\ldots\times n_k}$ is the subset of $X^{(n_1)} \times \cdots \times X^{(n_k)} \times X$ defined by

$$Z_{n_1\times\ldots\times n_k} = \Big\{(\underline{x}_1, \ldots, \underline{x}_k, x) \text{ in } X^{(n_1)} \times \cdots \times X^{(n_k)} \times X \\ \text{such that } x \in \mathrm{supp}(\underline{x}_1 \cup \cdots \cup \underline{x}_k)\Big\}\,.$$

2.2. Hilbert–Douady schemes

- If Y is a smooth complex surface and n is a positive integer, $Y^{[n]}$ is the Hilbert–Douady scheme of n–points in Y (i.e. the moduli space parameterizing zero-dimensional subschemes of Y of length n); $Y^{[n]}$ is smooth of dimension $2n$ and is irreducible if Y is irreducible [3,8].
- The Hilbert–Chow morphism from $Y^{[n]}$ to $Y^{(n)}$ (also called Douady-Barlet morphism in the analytic setting) is denoted by Γ.
- If E is a holomorphic vector bundle on Y, $E^{[n]}$ denotes the n–th tautological vector bundle on Y; it satisfies that for any ξ in $Y^{[n]}$, $E^{[n]}_{|\xi} = H^0(\xi, E)$.
- If n, n' are positive integers such that $n' > n$, the *incidence variety* $Y^{[n',\,n]}$ (in the notation of [16]) is the set of pairs (ξ, ξ') in $X^{[n]} \times X^{[n']}$ such that ξ is a subscheme of ξ'. Remark that some authors (e.g. [7]) use the other possible notation $Y^{[n,\,n']}$.

2.3. Morphisms

- Let $\pi : Y \longrightarrow Z$ be a given morphism. For any z in Z, the fiber $\pi^{-1}(z)$ of z is denoted by Y_z.
- Let $\pi : Y \longrightarrow Z$, $\pi' : Y' \longrightarrow Z$ and $f : Y \longrightarrow Y'$ be three morphisms such that $\pi' \circ f = \pi$. For any z in Z, the restriction of f to the fiber Y_b is denoted by f_b; it is a morphism from Y_b to Y'_b.
- If $f : Y \longrightarrow Z$ is a morphism, then for any positive integer n, f induces a morphism $f^{(n)} : Y^{(n)} \longrightarrow Z^{(n)}$. If Y and Z are complex surfaces and if f is a biholomorphism, f induces a biholomorphism $f_* : Y^{[n]} \longrightarrow Z^{[n]}$ given at the level of ideal sheaves by the composition with f^{-1}.

2.4. Relative integrable complex structures

- Let $k, n_1, \dots, n_k$ be positive integers and W be a neighbourhood of the incidence set $Z_{n_1 \times \cdots \times n_k}$ in $X^{(n_1)} \times \cdots \times X^{(n_k)} \times X$ (see Sect. 2.1). A *relative integrable complex structure* J^{rel} on W is a smooth family of integrable complex structures on the fibers of $\text{pr}_1 : W \longrightarrow X^{(n_1)} \times \cdots \times X^{(n_k)}$. In other words, for any element $(\underline{x}_1, \dots, \underline{x}_k)$ in $X^{(n_1)} \times \cdots \times X^{(n_k)}$, $J^{\text{rel}}_{\underline{x}_1, \dots, \underline{x}_k}$ is an integrable complex structure on $W_{\underline{x}_1, \dots, \underline{x}_k}$ varying smoothly with the $\underline{x}_j$'s.
- We use in a systematic way the following notation throughout the article: a relative integrable structure in a neighbourhood of $Z_{n_1 \times \cdots \times n_k}$ is identified by an index "$n_1 \times \cdots \times n_k$". Thus, the occurrence of such a term as $J^{\text{rel}}_{n_1 \times \cdots \times n_k, \underline{x}_1, \dots, \underline{x}_k}$ means that:
  - * $J^{\text{rel}}_{n_1 \times \cdots \times n_k}$ is a relative integrable complex structure in a neighbourhood of $Z_{n_1 \times \cdots \times n_k}$ in $X^{(n_1)} \times \cdots \times X^{(n_k)} \times X$.
  - * $\underline{x}_1 \in X^{(n_1)}, \dots, \underline{x}_k \in X^{(n_k)}$.

  In this case, $J^{\text{rel}}_{n_1 \times \cdots \times n_k, \underline{x}_1, \dots, \underline{x}_k}$ is an integrable complex structure in a neighbourhood of $\text{supp}(\underline{x}_1 \cup \cdots \cup \underline{x}_k)$ in X. When $k = 1$, we use the notation J^{rel} instead of J^{rel}_n if no confusion is possible.
- We deal several times with compatibility conditions between relative integrable complex structures parameterized by different products of symmetric powers of X. This is possible because relative integrable complex structures are families of integrable complex structures on open subsets of the *same* manifold X, even if the parameter spaces can be different.
- Let g be a Riemannian metric on X, $n_1, \dots, n_k$ be positive integers and $J^{\text{rel}}_{n_1 \times \cdots \times n_k}$ be a relative integrable complex structure in a neighbourhood W of $Z_{n_1 \times \cdots \times n_k}$. We define

$$\left\| J^{\text{rel}}_{n_1 \times \cdots \times n_k} - J \right\|_{C^0,\, g,\, W} = \sup_{\substack{(\underline{x}_1, \dots, \underline{x}_k) \in X^{(n_1)} \times \cdots X^{(n_k)} \\ p \in W_{\underline{x}_1, \dots, \underline{x}_k}}} \left\| J^{\text{rel}}_{n_1 \times \cdots \times n_k, \underline{x}_1, \dots, \underline{x}_k}(p) - J(p) \right\|_{\widetilde{g}(p)}$$

where $\widetilde{g}$ is the Riemannian metric on $\mathrm{End}(TX)$ associated with g. The relative integrable complex structure $J^{\,\mathrm{rel}}_{n_1\times\cdots\times n_k}$ is said to be *close to* J if $\left\| J^{\,\mathrm{rel}}_{n_1\times\cdots\times n_k} - J \right\|_{C^0,\, g, W}$ is sufficiently small.

3. The Hilbert schemes of an almost-complex compact four-manifold

3.1. Voisin's construction

In this section, we recall Voisin's construction of the almost-complex Hilbert scheme and establish some complementary results. We use the notations and the terminology introduced in Sect. 2.

Definition 1. If g is a Riemannian metric on X and if ε is a positive real number, let $\mathcal{B}_{g,\varepsilon}$ be the set of pairs $(W, J^{\,\mathrm{rel}})$ such that W is a neighbourhood of the incidence set Z_n in $X^{(n)} \times X$, $J^{\,\mathrm{rel}}$ is a relative integrable complex structure on W and $\| J^{\,\mathrm{rel}} - J \|_{C^0,\, g,\, W} < \varepsilon$.

For ε small enough, $\mathcal{B}_{g,\varepsilon}$ is connected and weakly contractible, i.e. $\pi_i(\mathcal{B}_\varepsilon) = 0$ for $i \geq 1$ [12]. We choose such an ε and write $\mathcal{B}$ instead of $\mathcal{B}_{g,\varepsilon}$.

Let $\pi: (W^{[n]}_{\mathrm{rel}}, J^{\,\mathrm{rel}}) \longrightarrow X^{(n)}$ be the relative Hilbert scheme of $(W, J^{\,\mathrm{rel}})$ over $X^{(n)}$; the fibers of π are the smooth analytic sets $(W^{[n]}_{\underline{x}}, J^{\,\mathrm{rel}}_{\underline{x}})$, $\underline{x} \in X^{(n)}$. We denote by $\Gamma_{\mathrm{rel}}: W^{[n]}_{\mathrm{rel}} \longrightarrow W^{(n)}_{\mathrm{rel}}$ the associated relative Hilbert–Chow morphism over $X^{(n)}$.

Definition 2. The *topological Hilbert scheme* $X^{[n]}_{J^{\,\mathrm{rel}}}$ is defined by

$$X^{[n]}_{J^{\,\mathrm{rel}}} = (\pi, \mathrm{pr}_2 \circ \Gamma_{\mathrm{rel}})^{-1} \left(\Delta_{X^{(n)}} \right),$$

where $\Delta_{X^{(n)}}$ is the diagonal of $X^{(n)}$ in $X^{(n)} \times X^{(n)}$. More explicitly,

$$X^{[n]}_{J^{\,\mathrm{rel}}} = \{ (\xi, \underline{x}) \text{ such that } \underline{x} \in X^{(n)},\ \xi \in (W^{[n]}_{\underline{x}}, J^{\,\mathrm{rel}}_{\underline{x}}) \text{ and } \Gamma(\xi) = \underline{x} \}\cdot$$

To put a differentiable structure on $X^{[n]}_{J^{\,\mathrm{rel}}}$, Voisin uses specific relative integrable structures which are invariant by a compatible system of retractions on the strata of $X^{(n)}$. These relative structures are differentiable for a differentiable structure $\mathfrak{D}_J$ on $X^{(n)}$ which depends on J (and on other additional data) and is weaker than the quotient differentiable structure, i.e. $\mathfrak{D}_J \subseteq \mathcal{C}^\infty_{X^{(n)}}$. Voisin's main results are the following ones:

Theorem 3. [20,21]

(i) $X^{[n]}$ *is a* $4n$*–dimensional differentiable manifold, well-defined modulo diffeomorphisms isotopic to the identity.*
(ii) The Hilbert–Chow map $\Gamma: X^{[n]} \longrightarrow \left(X^{(n)}, \mathfrak{D}_J \right)$ *is differentiable and its fibers* $\Gamma^{-1}(\underline{x})$ *are homeomorphic to the fibers of the usual Hilbert–Chow morphism for any integrable structure near* $\mathrm{supp}(\underline{x})$.
(iii) $X^{[n]}$ *can be endowed with a stable almost-complex structure, hence defines an almost-complex cobordism class. When X is symplectic,* $X^{[n]}_{J^{\,\mathrm{rel}}}$ *is symplectic.*

The first point is the analogue of Fogarty's result [8] in the differentiable case. In this article we do not use differentiable properties of $X^{[n]}$ but only topological ones, which allow us to work with $X^{[n]}_{J^{\,\mathrm{rel}}}$ for any $J^{\,\mathrm{rel}}$ in $\mathcal{B}$. Without any assumption on $J^{\,\mathrm{rel}}$, the first point of Theorem 3 has the following topological version:

Proposition 1. *For any $J^{\,\mathrm{rel}}$ in $\mathcal{B}$, $X^{[n]}_{J^{\,\mathrm{rel}}}$ is a topological manifold of dimension* $4n$.

Proof. Let W be the neighbourhood of Z_n associated with $J^{\,\mathrm{rel}}$, $\underline{x}_0$ be any element in $X^{(n)}$ and x_0 be a lift of $\underline{x}_0$ in X^n. We write $x_0 = (y_1, \ldots, y_1, \ldots, y_k, \ldots, y_k)$ where the points y_j are pairwise distinct for $1 \leq j \leq k$ and each y_j appears n_j times. If Ω is a small neighbourhood of $\mathrm{supp}(\underline{x}_0)$ in X, we can assume that Ω is an open subset of $\mathbb{C}^2$. For $\epsilon > 0$ small enough, the balls $B(y_j, \varepsilon)$ are contained in Ω and are also pairwise disjoint in X. If $H = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$ is the stabilizer of x_0 in $\mathfrak{S}_n$, we put $V_{x_0} = B(y_1, \varepsilon)^{n_1} \times \cdots \times B(y_k, \varepsilon)^{n_k}$. Then $V_{\underline{x}_0} = V_{x_0}/H$ is a neighbourhood of $\underline{x}_0$ in $X^{[n]}$. If $U = \cup_{j=1}^{k} B(y_j, \varepsilon)$, a sufficiently small choice of ε guarantees that for every $\underline{x}$ in $V_{\underline{x}_0}$, $U \subseteq W_{\underline{x}}$. By the Newlander–Nirenberg theorem with parameters, there exists a smooth map $\phi\colon\ V_{x_0} \times U \longrightarrow \mathbb{C}^2$ invariant under H such that for any x in V_{x_0}, if $\underline{x}$ is the image of x in $V_{\underline{x}_0}$, then $\phi(x, .)$ is a biholomorphism between $(U, J^{\mathrm{rel}}_{\underline{x}})$ and its image in $\mathbb{C}^2$ endowed with the standard complex structure; we can even assume that $\phi(x_0, \,.\,) = \mathrm{id}$.

If we put $\phi(x, p) = (z_{\underline{x}}(p), w_{\underline{x}}(p))$, this means that $(z_{\underline{x}}, w_{\underline{x}})$ are relative holomorphic coordinates on $(U, J^{\mathrm{rel}}_{\underline{x}})$. Let us introduce new holomorphic coordinates on U: if $D_1\phi$ is the partial differential of ϕ with respect to the variable x in V_{x_0}, we define a function $\widetilde{\phi}\colon\ V_{x_0} \times U \longrightarrow \mathbb{C}^2$ as follows: for $1 \leq j \leq k$ and p in $B(y_j, \varepsilon)$, $\widetilde{\phi}(x, p) = \phi(x, p) - D_1\,\phi(x_0, y_j)(x - x_0)$. If

$$\Psi:\ B(y_1, \varepsilon)^{n_1} \times \cdots \times B(y_k, \varepsilon)^{n_k} \longrightarrow \left(\mathbb{C}^2\right)^n$$

is defined by

$$\Psi(x_1, \ldots, x_n) = (\widetilde{\phi}(x_1, \ldots, x_n, x_1), \ldots, \widetilde{\phi}(x_1, \ldots, x_n, x_n)),$$

then Ψ is H–equivariant and its differential at x_0 is the identity map, so that Ψ induces a local homeomorphism ψ of $(\mathbb{C}^2)^{(n)}$ with itself around $\underline{x}_0$. We can now construct a topological chart $\varphi\colon\ \Gamma^{-1}(V_{\underline{x}_0}) \longrightarrow \Gamma^{-1}(\psi(V_{\underline{x}_0}))$ on $X^{[n]}_{J^{\,\mathrm{rel}}}$: for ξ in $\Gamma^{-1}(V_{\underline{x}_0})$, if $\underline{x} = \Gamma(\xi)$, $\varphi(\xi) = \widetilde{\phi}(\underline{x}, \,.\,)_*\xi$. The inverse of φ is given for any η in $\psi(V_{\underline{x}_0})^{[n]}$ by the formula $\varphi^{-1}(\eta) = (\widetilde{\phi}(\underline{y}, \,.\,)^{-1})_*\eta$, where $\underline{y} = \psi^{-1}(\Gamma(\eta))$. □

Remark 1. Let J_0^{rel} and J_1^{rel} be two elements in $\mathcal{B}$ and fix $\underline{x}_0$ in $X^{(n)}$. We use the notations of the preceding proof. If $W_{\underline{x}_0}$ is a small neighbourhood of $\underline{x}_0$ in $(\mathbb{C}^2)^{(n)}$, we can assume that $W_{\underline{x}_0} \subseteq V_{0,\underline{x}_0}$, and that $\psi_0(W_{\underline{x}_0}) \subseteq \psi_1(V_{1,\underline{x}_0})$. Besides, we can choose ε_0 and ε_1 in order that $\widetilde{\phi}_0(V_{0,\underline{x}_0} \times U_0) \subseteq \widetilde{\phi}_1(V_{1,\underline{x}_0} \times U_1)$. If we define two functions $\widehat{\psi}\colon\ V_{0,\underline{x}_0} \longrightarrow V_{1,\underline{x}_0}$ and $\widehat{\phi}\colon\ V_{0,\underline{x}_0} \times U_0 \longrightarrow V_{1,\underline{x}_0} \times U_1$ by

$\widehat{\psi}(\underline{x}) = \psi_1^{-1}\psi_0(\underline{x})$ and $\widehat{\phi}(\underline{x}, p) = \widetilde{\phi}_1^{-1}(\psi(\underline{x}), \widetilde{\phi}_0(\underline{x}, p))$, then we obtain a commutative diagram

$$\begin{array}{ccccccc}
X^{[n]}_{J_0^{\mathrm{rel}}} & \hookleftarrow & \Gamma^{-1}(W_{\underline{x}_0}) & \xrightarrow[\sim]{\widehat{\phi}_*} & \Gamma^{-1}(\widehat{\psi}(W_{\underline{x}_0})) & \hookrightarrow & X^{[n]}_{J_1^{\mathrm{rel}}} \\
\downarrow{\scriptstyle \Gamma} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Gamma} \\
X^{(n)} & \supseteq & W_{\underline{x}_0} & \xrightarrow[\sim]{\widehat{\psi}} & \widehat{\psi}(W_{\underline{x}_0}) & \subseteq & X^{(n)}
\end{array}$$

and $\widehat{\psi}$ is a stratified isomorphism. This proves that $X^{[n]}_{J_0^{\mathrm{rel}}}$ and $X^{[n]}_{J_1^{\mathrm{rel}}}$ are locally homeomorphic over a neighbourhood of $\underline{x}_0$.

3.2. Göttsche's formula

We now turn our attention to the cohomology of $X^{[n]}_{J^{\mathrm{rel}}}$. The first step is the computation of the Betti numbers of $X^{[n]}$. We first recall the proof of Göttsche and Soergel [11] for projective surfaces, and then we show how to adapt it in the non-integrable case.

Let Y and Z be irreducible algebraic complex varieties and $f : Y \longrightarrow Z$ be a proper morphism. We assume that Z is stratified in such a way that f is a topological fibration over each stratum Z_ν. If $Y_\nu = f^{-1}(Z_\nu)$, let d_ν be the real dimension of the largest irreducible component of Y_ν. Then $R^{d_\nu} f_* \mathbb{Q}_{Y_\nu}$ is the associated monodromy local system on Z_ν, we denote it by $\mathcal{L}_\nu$.

Definition 3. – The map f is *semi-small* if for all ν, $\mathrm{codim}_Z Z_\nu \geq d_\nu$.
– If f is semi-small, a stratum Z_ν is *relevant* if $\mathrm{codim}_Z Z_\nu = d_\nu$.

Theorem 4. (Decomposition theorem for semi-small maps [2]) *If Y is rationally smooth and $f : Y \longrightarrow Z$ is a proper semi-small morphism, there exists a canonical quasi-isomorphism*

$$Rf_*\mathbb{Q}_Y \xrightarrow{\ \sim\ } \bigoplus_{\nu \text{ relevant}} j_{\nu *}\, IC^{\bullet}_{\overline{Z}_\nu}(\mathcal{L}_\nu)[-d_\nu]$$

in the bounded derived category of $\mathbb{Q}$–constructible sheaves on Z, where $IC^{\bullet}_{\overline{Z}_\nu}(\mathcal{L}_\nu)$ is the intersection complex on $\overline{Z}_\nu$ associated to the monodromy local system $\mathcal{L}_\nu$ and $j_\nu : \overline{Z}_\nu \longrightarrow Z$ is the inclusion. In particular,

$$H^k(Y, \mathbb{Q}) = \bigoplus_{\nu \text{ relevant}} IH^{k-d_\nu}(\overline{Z}_\nu, \mathcal{L}_\nu).$$

Remark 2. A topological proof of Theorem 4 can be found in the unpublished lecture notes [15], we reproduce it in Appendix (Sect. 5.2). This proof shows that there exists a canonical quasi-isomorphism

$$Rf_*\mathbb{Q}_Y \simeq \bigoplus_{\nu \text{ relevant}} j_{\nu *}\, IC^{\bullet}_{\overline{Z}_\nu}(\mathcal{L}_\nu)[-d_\nu]$$

under the following weaker topological hypotheses: Y is a rationally smooth connected topological space (which means that the dualizing complex ω_Y of Y with rational coefficients is a local system concentrated in a single degree), Z is a stratified topological space and $f\colon Y \longrightarrow Z$ is a proper continuous map which is locally homeomorphic over Z (in a stratified way) to a semi-small map between complex algebraic varieties. This is the key of the proof of Theorem 5 below.

If X is a quasi-projective surface, the Hilbert–Chow morphism is semi-small with irreducible fibers [3], so that the monodromy local systems are trivial; and $X^{[n]}$ is smooth. Then Göttsche's formula for the generating series of the Betti numbers $b_i(X^{[n]})$ follows directly from the decomposition theorem. We now extend this result for almost-complex Hilbert schemes.

Theorem 5. (Göttsche's formula) *If (X, J) is an almost-complex compact four-manifold, then for any integrable complex structure $J^{\,\mathrm{rel}}$,*

$$\sum_{n=0}^{\infty}\sum_{i=0}^{4n} b_i\left(X^{[n]}_{J^{\,\mathrm{rel}}}\right) t^i q^n = \prod_{m=1}^{\infty} \frac{\left[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\right]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

Proof. The topological charts on $X^{[n]}_{J^{\,\mathrm{rel}}}$ constructed in Proposition 1 show that the Hilbert-Chow maps $\Gamma\colon X^{[n]}_{J^{\,\mathrm{rel}}} \longrightarrow X^{(n)}$ and $\Gamma\colon (\mathbb{C}^2)^{[n]} \longrightarrow (\mathbb{C}^2)^{(n)}$ are locally homeomorphic. The latter map being a semi-small morphism, the decomposition theorem applies by Remark 2; and the computations are the same as in the integrable case (see [18, Sect. 6.2]). □

3.3. Variation of the relative integrable structure

Theorem 5 implies in particular that the Betti numbers of $X^{[n]}_{J^{\,\mathrm{rel}}}$ are independent of $J^{\,\mathrm{rel}}$. We now prove a stronger result, namely that the Hilbert schemes corresponding to different relative integrable complex structures are homeomorphic.

Proposition 2. *(i) Let $\left(J_t^{\,\mathrm{rel}}\right)_{t\in B(0,r)\subseteq\mathbb{R}^d}$ be a smooth path in $\mathcal{B}$. Then the associated relative Hilbert scheme $\left(X^{[n]}, \left\{J_t^{\,\mathrm{rel}}\right\}_{t\in B(0,r)}\right)$ over $B(0,r)$ is a topological fibration.*

(ii) If $J_0^{\,\mathrm{rel}}$, $J_1^{\,\mathrm{rel}}$ are two elements of $\mathcal{B}$, then there exist canonical isomorphisms

$$H^*\left(X^{[n]}_{J_0^{\,\mathrm{rel}}}, \mathbb{Q}\right) \simeq H^*\left(X^{[n]}_{J_1^{\,\mathrm{rel}}}, \mathbb{Q}\right) \quad \text{and} \quad K\left(X^{[n]}_{J_0^{\,\mathrm{rel}}}\right) \simeq K\left(X^{[n]}_{J_1^{\,\mathrm{rel}}}\right).$$

In order to prove Proposition 2, we start with a technical result:

Proposition 3. *Let $\left(J_t^{\,\mathrm{rel}}\right)_{t\in B(0,r)\subseteq\mathbb{R}^d}$ be a smooth family of relative integrable complex structures in a neighbourhood of Z_n varying smoothly with t. Then there exist a positive real number ε, a neighbourhood W of Z_n in $X^{(n)}\times X$ and a smooth map $\psi\colon (t, \underline{x}, p) \longmapsto \psi_{t,\underline{x}}(p)$ from $B(0,\varepsilon)\times W$ to X such that:*

(i) $\psi_{0,\underline{x}}(p) = p$,

(ii) *For any couple* $(t, \underline{x})$ *in* $B(0, \varepsilon) \times X^{(n)}$, $\psi_{t,\underline{x}}$ *is a biholomorphism between* $W_{\underline{x}}$ *and its image, these two open subsets of* X *being endowed with the integrable complex structures* $J^{\text{rel}}_{0,\underline{x}}$ *and* $J^{\text{rel}}_{t,\psi^{(n)}_{t,\underline{x}}(\underline{x})}$ *respectively.*

(iii) *For all* t *in* $B(0, \varepsilon)$*, the map* $\underline{x} \longmapsto \psi^{(n)}_{t,\underline{x}}(\underline{x})$ *is a homeomorphism of* $X^{(n)}$ *with itself.*

Proof. We can find two neighbourhoods W' and W of Z_n in $X^{(n)} \times X$ as well as a smooth map $\theta_t : (t, \underline{x}, p) \longrightarrow \theta_{t,x}(p)$ from $B(0, r) \times W'$ to X such that for any (t, x) in $B(0, r) \times W'$, the conditions below are satisfied:

– $W_{\underline{x}} \subseteq \theta_{t,\underline{x}}(W'_{\underline{x}}) \cap W'_{\underline{x}}$.

– $\theta_{t,\underline{x}} : \left(W'_{\underline{x}}, J^{\text{rel}}_{t,\theta_{t,\underline{x}}(\underline{x})}\right) \longrightarrow \left(\phi_{t,\underline{x}}(W'_{\underline{x}}), J^{\text{rel}}_{0,\psi_{t,\underline{x}}(\underline{x})}\right)$ is a biholomorphism.

– If $t = 0$, $\theta_{0,\underline{x}} = \text{id}$.

Let us take a covering $\left(U_i\right)_{1 \le i \le N}$ of $X^{(n)}$ and relative holomorphic coordinates on W' above each U_i given by maps $\phi_i : W' \cap pr_1^{-1}(U_i) \longrightarrow \mathbb{C}^2$ such that for each i, the map $\underline{x} \longmapsto \phi^{(n)}_{i,\underline{x}}(\underline{x})$ is a homeomorphism between U_i and its image V_i in $(\mathbb{C}^2)^{(n)}$ (see the proof of Proposition 1). Then we define relative holomorphic coordinates $\left(\phi_{i,t,\underline{x}}\right)_{1 \le i \le N}$ for the relative integrable complex structure J^{rel}_t on $W \cap pr_1^{-1}(U_i)$ by the formula $\phi_{i,t,\underline{x}}(p) = \phi_{i,\underline{x}}\left(\theta_{t,\underline{x}}(p)\right)$. For small t, after shrinking U_i if necessary, the map $\underline{x} \longmapsto \phi^{(n)}_{i,t,\underline{x}}(\underline{x})$ is still a homeomorphism of U_i with its image: indeed the map $\underline{x} \longmapsto \phi^{(n)}_{i,t,\underline{x}}(\underline{x})$ is obtained from the $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$–equivariant smooth map

$$(x_1, \ldots, x_n) \longrightarrow \left(\phi_t(x_1, \ldots, x_n, x_1), \ldots, \phi_t(x_1, \ldots, x_n, x_n)\right)$$

and we use the fact that a sufficiently small smooth perturbation of a smooth diffeomorphism remains a smooth diffeomorphism. If $\widetilde{Z}_n \subseteq (\mathbb{C}^2)^{(n)} \times \mathbb{C}^2$ is the incidence set of $\mathbb{C}^2$, the map $\check{\phi}_{i,t} : (\underline{x}, p) \longmapsto \left(\phi^{(n)}_{i,t,\underline{x}}(\underline{x}), \phi_{i,t,\underline{x}}(p)\right)$ is a homeomorphism between $W \cap pr_1^{-1}(U_i)$ and its image, which is a neighbourhood of $\widetilde{Z}_n$ over V_i.

Let us introduce the following notation: for any map φ defined on an open set Ω of $X^{(n)} \times X$ with values in X or in $\mathbb{C}^2$, we denote by $\check{\varphi}$ the map from Ω to $X^{(n)} \times X$ given by $(\underline{x}, p) \longmapsto (\varphi^{(n)}_{t,\underline{x}}(\underline{x}), \varphi_{t,\underline{x}}(p))$. Then the condition (ii) of the proposition means that $\check{\phi}_{i,t} \circ \check{\psi}_t \circ \left(\check{\phi}_{i,0}\right)^{-1} = \check{u}_t$, where $(\check{u}_t)_{t \in B(0,\varepsilon)}$ is a smooth family of smooth maps from $\check{\phi}_{i,0}(U_i)$ to $\mathbb{C}^2$ such that for every $\underline{y}$ in V_i, $u_{t,\underline{y}}$ is a biholomorphism between $\check{\phi}_{i,0}(U_i)_{\underline{y}}$ and its image, both being endowed with the standard complex structure of $\mathbb{C}^2$. The condition (i) means that $u_{0,\,\underline{y}} = \mathrm{id}$. Thus $\left(\psi_t\right)_{||t||\leq\varepsilon}$ can be constructed on each open set U_i (it suffices to choose $u_{t,\,\underline{y}} = \mathrm{id}$).

Since biholomorphisms close to the identity form a contractible set, we can, using cut-off functions, glue together the local solutions on $X^{(n)}$ to obtain a global one. The map $\underline{x} \longrightarrow \psi^{(n)}_{t,\,\underline{x}}(\underline{x})$ is induced by a smooth $\mathfrak{S}_n$–equivariant map of X^n into X^n (and is a small perturbation of the identity map if $||t||$ is small enough), thus a $\mathfrak{S}_n$–equivariant diffeomorphism of X^n. We have therefore defined a family of maps $\left(\psi_t\right)_{||t||\leq\varepsilon}$ satisfying the conditions (i), (ii) and (iii). □

We can now prove Proposition 2.

Proof of Proposition 2. (i) We have

$$\left(X^{[n]}, \left\{J_t^{\,\mathrm{rel}}\right\}_{t\in B(0,r)}\right) = \Big\{(\xi, \underline{x}, t) \text{ such that } \underline{x} \in X^{(n)},\ t \in B(0, r),\ \xi \in \left(W^{[n]}_{\underline{x}}, J^{\,\mathrm{rel}}_{t,\,\underline{x}}\right) \text{ and } \Gamma(\xi) = \underline{x}\Big\}.$$

A topological trivialization of this family over $B(0, r)$ near zero is given by the map

$$\Phi:\ X^{[n]}_{J_0^{\,\mathrm{rel}}} \times B(0, \varepsilon) \longrightarrow \left(X^{[n]}, \left\{J_t^{\,\mathrm{rel}}\right\}_{t\in B(0,\varepsilon)}\right)$$

defined by $\Phi(\xi, \underline{x}, t) = \left(\psi_{t,\,\underline{x}*}\xi,\ \psi_{t,\underline{x}}(\underline{x}), t\right)$, where ψ is given by Proposition 3. This proves that the relative Hilbert scheme is locally topologically trivial over $B(0, r)$.

(ii) The set $\mathcal{B}$ being connected, point (i) shows that $X^{[n]}_{J_0^{\,\mathrm{rel}}}$ and $X^{[n]}_{J_1^{\,\mathrm{rel}}}$ are homeomorphic. Let us consider two smooth paths $\left(J^{\,\mathrm{rel}}_{0,\,t}\right)_{0\leq t\leq 1}$ and $\left(J^{\,\mathrm{rel}}_{1,\,t}\right)_{0\leq t\leq 1}$ between $J_0^{\,\mathrm{rel}}$ and $J_1^{\,\mathrm{rel}}$. Since $\pi_1(\mathcal{B}) = 0$, we can find a smooth family $\left(J^{\,\mathrm{rel}}_{s,\,t}\right)_{\substack{0\leq s\leq 1\\ 0\leq t\leq 1}}$ which is an homotopy between the two paths. The associated relative Hilbert scheme over $[0, 1] \times [0, 1]$ is locally topologically trivial, hence globally trivial since $[0, 1]\times[0, 1]$ is contractible. This shows that the homeomorphisms between $X^{[n]}_{J_0^{\,\mathrm{rel}}}$ and $X^{[n]}_{J_1^{\,\mathrm{rel}}}$ constructed by choosing a path between $J_0^{\,\mathrm{rel}}$ and $J_1^{\,\mathrm{rel}}$ and taking a topological trivialization of the relative Hilbert scheme associated with this path belong to a canonical homotopy class. □

As a consequence of this proposition, there exists a ring $H^*(X^{[n]}, \mathbb{Q})$ (resp. $K(X^{[n]})$) such that for any $J^{\,\mathrm{rel}}$ close to J, $H^*(X^{[n]}, \mathbb{Q})$ (resp. $K(X^{[n]})$) and $H^*\left(X^{[n]}_{J^{\,\mathrm{rel}}}, \mathbb{Q}\right)$ (resp. $K\left(X^{[n]}_{J^{\,\mathrm{rel}}}\right)$) are canonically isomorphic.

In the sequel, we will deal with products of Hilbert schemes. We could of course consider products of the type $X^{[n]}_{J_n^{\,\mathrm{rel}}} \times X^{[m]}_{J_m^{\,\mathrm{rel}}}$, but it is necessary in practice to work with pairs of relative integrable complex structures parameterized by elements in the product $X^{(n)} \times X^{(m)}$. Let W be a small neighbourhood of $Z_{n\times m}$ (see Sect. 2.1) in $X^{(n)} \times X^{(m)} \times X$ and let $J^{1,\mathrm{rel}}$ and $J^{2,\mathrm{rel}}$ be two relative integrable complex structures on the fibers of $\mathrm{pr}_1 : W \longrightarrow X^{(n)} \times X^{(m)}$.

Definition 4. The product Hilbert scheme $\left(X^{[n]\times[m]}, J^{1,\mathrm{rel}}, J^{2,\mathrm{rel}}\right)$ is defined by

$$\left(X^{[n]\times[m]}, J^{1,\mathrm{rel}}, J^{2,\mathrm{rel}}\right) = \Big\{(\xi, \eta, \underline{x}, \underline{y}) \text{ such that } \underline{x} \in X^{(n)},\ \underline{y} \in X^{(m)},$$
$$\xi \in \left(W^{[n]}_{\underline{x},\underline{y}}, J^{1,\mathrm{rel}}_{\underline{x},\underline{y}}\right),\ \eta \in \left(W^{[m]}_{\underline{x},\underline{y}}, J^{2,\mathrm{rel}}_{\underline{x},\underline{y}}\right),\ \Gamma(\xi) = \underline{x} \text{ and } \Gamma(\eta) = \underline{y}\Big\} \cdot$$

The same definition holds for products of the type

$$\left(X^{[n_1]\times\cdots\times[n_k]}, J^{1,\mathrm{rel}}, \ldots, J^{k,\mathrm{rel}}\right).$$

If there exist two relative integrable complex structures J_n^{rel} and J_m^{rel} in neighbourhoods of Z_n and Z_m such that for all $\underline{x}$ in $X^{(n)}$ and all $\underline{y}$ in $X^{(m)}$, $J^{1,\mathrm{rel}}_{\underline{x},\underline{y}} = J^{\mathrm{rel}}_{n,\underline{x}}$ and $J^{2,\mathrm{rel}}_{\underline{x},\underline{y}} = J^{\mathrm{rel}}_{m,\underline{y}}$ in small neighbourhoods of $\mathrm{supp}(\underline{x})$ and $\mathrm{supp}(\underline{y})$ respectively, then

$$\left(X^{[n]\times[m]}, J^{1,\mathrm{rel}}, J^{2,\mathrm{rel}}\right) = X^{[n]}_{J_n^{\mathrm{rel}}} \times X^{[m]}_{J_m^{\mathrm{rel}}}.$$

If $\left(J_t^{1,\mathrm{rel}}, J_t^{2,\mathrm{rel}}\right)_{t\in B(0,r)}$ is a smooth family of relative integrable complex structures, it can be shown as in Propositions 2 and 3 that the relative product Hilbert scheme $\left(X^{[n]\times[m]}, \left\{J_t^{1,\mathrm{rel}}\right\}_{t\in B(0,r)}, \left\{J_t^{2,\mathrm{rel}}\right\}_{t\in B(0,r)}\right)$ is topologically trivial over $B(0, r)$. Thus the product Hilbert schemes $\left(X^{[n]\times[m]}, J^{1,\mathrm{rel}}, J^{2,\mathrm{rel}}\right)$ are homeomorphic to products $X^{[n]}_{J_n^{\mathrm{rel}}} \times X^{[m]}_{J_m^{\mathrm{rel}}}$ of usual Hilbert schemes. If the structures $J^{1,\mathrm{rel}}$ and $J^{2,\mathrm{rel}}$ are equal, then $\left(X^{[n]\times[m]}, J^{1,\mathrm{rel}}, J^{2,\mathrm{rel}}\right)$ consists of pairs of schemes of given support (parameterized by $X^{(n)} \times X^{(m)}$) for the *same* integrable structure. These product Hilbert schemes are therefore well adapted for the study of incidence relations.

4. Incidence varieties and Nakajima operators

4.1. Construction of incidence varieties

If J is an integrable complex structure on X, the *incidence variety* $X^{[n',n]}$ defined in Sect. 2.2 is smooth for $n' = n + 1$ [19]. We have three maps

$$\lambda : X^{[n',n]} \longrightarrow X^{[n]}, \quad \nu : X^{[n',n]} \longrightarrow X^{[n']} \quad \text{and} \quad \rho : X^{[n',n]} \longrightarrow X^{(n'-n)}$$

given by $\lambda(\xi, \xi') = \xi$, $\nu(\xi, \xi') = \xi'$ and $\rho(\xi, \xi') = \mathrm{supp}\left(\mathcal{I}_\xi / \mathcal{I}_{\xi'}\right)$. Note that (λ, ν) is injective by definition.

If J is not integrable, we can define $X^{[n',n]}$ using the relative construction as explained below. For doing this, we choose a relative integrable complex structure $J^{\text{rel}}_{n\times(n'-n)}$ in a neighbourhood W of $Z_{n\times(n'-n)}$ in $X^{(n)}\times X^{(n'-n)}\times X$ (see Sect. 2.1).

Definition 5. The *incidence variety* $\left(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\right)$ is defined by

$$\begin{aligned}\left(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\right) = \Big\{&(\xi,\xi',\underline{x},\underline{y}) \text{ such that } \underline{x}\in X^{(n)},\ \underline{y}\in X^{(n'-n)},\\ &\xi\in\left(W^{[n]}_{\underline{x},\underline{y}}, J^{\text{rel}}_{n\times(n'-n),\,\underline{x},\,\underline{y}}\right),\ \xi'\in\left(W^{[n']}_{\underline{x},\underline{y}}, J^{\text{rel}}_{n\times(n'-n),\,\underline{x},\,\underline{y}}\right),\\ &\xi \text{ is a subscheme of } \xi',\ \Gamma(\xi)=\underline{x} \text{ and } \rho(\xi,\xi')=\underline{y}\Big\}\cdot\end{aligned}$$

Let $J^{\text{rel}}_{n\times n'}$ be a relative integrable complex structure in a neighbourhood of $Z_{n\times n'}$ such that for every $(\underline{u},\underline{v})$ in $X^{(n)}\times X^{(n'-n)}$, $J^{\text{rel}}_{n\times n',\,\underline{u},\,\underline{u}\cup\underline{v}} = J^{\text{rel}}_{n\times(n'-n),\,\underline{u},\,\underline{v}}$ in a neighbourhood of $\text{supp}(\underline{u}\cup\underline{v})$. Then

$$\left(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\right)\subseteq\left(X^{[n]\times[n']}, J^{\text{rel}}_{n\times n'}, J^{\text{rel}}_{n\times n'}\right).$$

If $\left\{J^{\text{rel}}_{t,\,n\times n'}\right\}_{t\in B(0,r)}$ is a smooth family of relative complex structures, we can take, as in Proposition 2, a topological trivialization of the family

$$\left(X^{[n]\times[n']}, \left\{J^{\text{rel}}_{t,\,n\times n'}\right\}_{t\in B(0,r)}, \left\{J^{\text{rel}}_{t,\,n\times n'}\right\}_{t\in B(0,r)}\right).$$

If we define $J^{\text{rel}}_{t,\,n\times(n'-n)}$ in a neighbourhood of $Z_{n\times(n'-n)}$ by the formula

$$J^{\text{rel}}_{t,n\times(n'-n),\,\underline{u},\,\underline{v}} = J^{\text{rel}}_{t,\,n\times n',\,\underline{u},\,\underline{u}\cup\underline{v}}\quad(\underline{u}\in X^{(n)},\,\underline{v}\in X^{(n'-n)}),$$

then the subfamily $\left(X^{[n',n]}, \left\{J^{\text{rel}}_{t,\,n\times(n'-n)}\right\}_{t\in B(0,r)}\right)$ is sent by the trivialization to a product $U^{[n',n]}\times B(0,\varepsilon)$, where U is an open set of $\mathbb{C}^2$. This means that the pair

$$\left\{\left(X^{[n',n]}, \left\{J^{\text{rel}}_{n\times(n'-n)}\right\}_{t\in B(0,r)}\right), \left(X^{[n]\times[n']}, \left\{J^{\text{rel}}_{n\times n'}\right\}_{t\in B(0,r)}, \left\{J^{\text{rel}}_{n\times n'}\right\}_{t\in B(0,r)}\right)\right\}$$

is locally, hence globally topologically trivial over $B(0,r)$.

The natural morphism from $\left(X^{[n',n]}, J_{n\times(n'-n)}\right)$ to $X^{(n)}\times X^{(n'-n)}$ is locally homeomorphic over $X^{(n)}\times X^{(n'-n)}$ to the morphism

$$(HC\circ\lambda,\rho)\colon\ (\mathbb{C}^2)^{[n',n]}\longrightarrow(\mathbb{C}^2)^{(n)}\times(\mathbb{C}^2)^{(n'-n)}.$$

This enables us to define a stratification on $X^{[n',n]}$ by patching together the analytic stratifications of a collection of $U_i^{[n',n]}$, where the U_i's are open subsets of $\mathbb{C}^2$. In this way, $\left(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\right)$ becomes a stratified CW–complex such that for each stratum S, $\dim(\overline{S}\backslash S)\leq\dim S-2$. In particular, each stratum has a fundamental homology class.

Let us introduce the following notations:

(i) The inverse image of the small diagonal of $X^{(n)}$ by $\Gamma\colon X^{[n]}_{J_n^{\mathrm{rel}}} \longrightarrow X^{(n)}$ is denoted by $\left(X_0^{[n]}, J_n^{\mathrm{rel}}\right)$.

(ii) The inverse image of the small diagonal of $X^{(n'-n)}$ by the residual map

$$\rho\colon \left(X^{[n',n]}, J^{\mathrm{rel}}_{n\times(n'-n)}\right) \longrightarrow X^{(n'-n)}$$

is denoted by $\left(X_0^{[n',n]}, J^{\mathrm{rel}}_{n\times(n'-n)}\right)$.

In the integrable case, $X_0^{[n',n]}$ is stratified by analytic sets $\left(Z_l\right)_{l\geq 0}$ defined by

$$Z_l = \left\{(\xi, \xi') \in X_0^{[n',n]} \text{ such that if } x = \rho(\xi, \xi'),\ \ell_x(\xi) = l\right\}; \tag{1}$$

Z_0 is irreducible of complex dimension $n' + n + 1$, and all the other Z_l have smaller dimensions (see [16]). By the same argument we have used to stratify almost-complex Hilbert schemes, this stratification also exists in the almost complex case. We prove the topological irreducibility of Z_0 in the following lemma:

Lemma 1. *Let* $\left[\,\overline{Z}_0\right]$ *be the fundamental homology class of* $\overline{Z}_0$. *Then*

$$H_{2(n'+n+1)}\left(X_0^{[n',n]}, J^{\mathrm{rel}}_{n\times(n'-n)}, \mathbb{Z}\right) = \mathbb{Z}\,.\left[\,\overline{Z}_0\right].$$

Proof. It is enough to prove that $H^{\mathrm{lf}}_{2(n'+n+1)}(Z_0, \mathbb{Z}) \simeq \mathbb{Z}$ (where H^{lf} denotes Borel-Moore homology), since all the remaining strata $\left(Z_l\right)_{l\geq 1}$ have dimensions smaller than $2(n' + n + 1) - 2$. Let

$$\begin{aligned}\widetilde{Z}_0 = \Big\{&(\xi, \eta, \underline{x}, p) \text{ such that } \underline{x} \in X^{(n)},\ \ p \in X,\\ &\xi \in \left(W^{[n]}_{\underline{x},\,(n'-n)p}, J^{\mathrm{rel}}_{n\times(n'-n),\,\underline{x},\,(n'-n)p}\right),\ \ \Gamma(\xi) = \underline{x},\\ &\eta \in \left(W^{[n'-n]}_{\underline{x},\,(n'-n)p}, J^{\mathrm{rel}}_{n\times(n'-n),\,\underline{x},\,(n'-n)p}\right) \text{ and } \Gamma(\eta) = (n'-n)p\Big\}.\end{aligned}$$

There is a natural inclusion $Z_0 \hookrightarrow \widetilde{Z}_0$ given by

$$\left(\xi, \xi', \underline{x}, (n'-n)p\right) \longmapsto (\xi, \xi'_{|p}, \underline{x}, p).$$

Remark that $\widetilde{Z}_0$ is compact. Since $\dim(\widetilde{Z}_0 \backslash Z_0) \leq 4n + 2(n' - n - 1)$, it suffices to show that $H_{2(n'+n+1)}(\widetilde{Z}_0, \mathbb{Z}) = \mathbb{Z}$. Besides, $\widetilde{Z}_0$ is a product-type Hilbert scheme homeomorphic to $X^{[n]}_{J_n^{\mathrm{rel}}} \times \left(X_0^{[n'-n]}, J^{\mathrm{rel}}_{n'-n}\right)$ for any relative integrable complex structures J^{rel}_n and $J^{\mathrm{rel}}_{n'-n}$. Since $\left(X_0^{[n'-n]}, J^{\mathrm{rel}}_{n'-n}\right)$ is, by Briançon's theorem [3], a topological fibration over X whose fiber is homeomorphic to an irreducible algebraic variety of complex dimension $n' - n - 1$, we obtain the result. □

4.2. Nakajima operators

We are now going to construct Nakajima operators $\mathfrak{q}_n(\alpha)$ for almost-complex four-manifolds. If $n' > n$ and if $J^{\text{rel}}_{n\times n'}$ is a relative integrable complex structure in a neighbourhood of $Z_{n\times n'}$, let us define

$$Q^{[n',n]} = \overline{Z}_0 \subseteq \left(X^{[n]\times[n']}, J^{\text{rel}}_{n\times n'}, J^{\text{rel}}_{n\times n'}\right) \times X, \qquad (2)$$

where the map on the last coordinate is given by the relative residual morphism and Z_0 is defined by (1). Since the pair $\left(Q^{[n',n]}, X^{[n]\times[n']} \times X\right)$ is topologically trivial when $J^{\text{rel}}_{n\times n'}$ varies, the cycle class $\left[Q^{[n',n]}\right]$ in $H_{2(n'+n+1)}\left(X^{[n]} \times X^{[n']} \times X, \mathbb{Z}\right)$ is independent of $J^{\text{rel}}_{n\times n'}$.

Definition 6. Let α be a rational cohomology class on X and j be a positive integer. We define the *Nakajima operators* $\mathfrak{q}_j(\alpha)$ and $\mathfrak{q}_{-j}(\alpha)$ as follows:

$$\begin{aligned}
\mathfrak{q}_j(\alpha):\ \bigoplus_{n\in\mathbb{N}} H^*(X^{[n]},\mathbb{Q}) &\longrightarrow \bigoplus_{n\in\mathbb{N}} H^*(X^{[n+j]},\mathbb{Q})\\
\tau &\longmapsto PD^{-1}\left[\mathrm{pr}_{2*}\left(\left[Q^{[n+j,n]}\right]\cap(\mathrm{pr}_3^*\,\alpha\cup\mathrm{pr}_1^*\,\tau)\right)\right]\\
\mathfrak{q}_{-j}(\alpha):\ \bigoplus_{n\in\mathbb{N}} H^*(X^{[n+j]},\mathbb{Q}) &\longrightarrow \bigoplus_{n\in\mathbb{N}} H^*(X^{[n]},\mathbb{Q})\\
\tau &\longmapsto PD^{-1}\left[\mathrm{pr}_{1*}\left(\left[Q^{[n+j,n]}\right]\cap(\mathrm{pr}_3^*\,\alpha\cup\mathrm{pr}_2^*\,\tau)\right)\right]
\end{aligned}$$

where pr_1, pr_2 and pr_3 are the projections from $X^{[n]} \times X^{[n+j]} \times X$ to each factor and PD is the Poincaré duality isomorphism between cohomology and homology. We also set $\mathfrak{q}_0(\alpha) = 0$

Remark 3. If α is a homogeneous rational cohomology class on X, let $|\alpha|$ denotes its degree. Then $q_j(\alpha)$ maps $H^i(X^{[n]},\mathbb{Q})$ to $H^{i+|\alpha|+2j-2}(X^{[n+j]},\mathbb{Q})$.

We now prove the following extension to the almost-complex case of Nakajima's relations [17]:

Theorem 6. *For all integers i, j and all homogeneous rational cohomology classes α and β on X, we have:*

$$\mathfrak{q}_i(\alpha)\mathfrak{q}_j(\beta) - (-1)^{|\alpha|\,|\beta|}\mathfrak{q}_j(\beta)\mathfrak{q}_i(\alpha) = i\,\delta_{i+j,0}\left(\int_X \alpha\beta\right)\mathrm{id}\,.$$

Proof. We adapt Nakajima's proof to our situation. Let us detail the most interesting case, which is the computation of $[\mathfrak{q}_{-i}(\alpha), \mathfrak{q}_j(\beta)]$ when i and j are positive. We introduce the classes P_{ij} (resp. Q_{ij}) in

$$H_*\left(X^{[n]},\mathbb{Q}\right)\otimes H_*\left(X^{[n-i]},\mathbb{Q}\right)\otimes H_*\left(X^{[n-i+j]},\mathbb{Q}\right)\otimes H_*(X,\mathbb{Q})\otimes H_*(X,\mathbb{Q}),$$

(resp.

$$H_*\left(X^{[n]},\mathbb{Q}\right)\otimes H_*\left(X^{[n+j]},\mathbb{Q}\right)\otimes H_*\left(X^{[n-i+j]},\mathbb{Q}\right)\otimes H_*(X,\mathbb{Q})\otimes H_*(X,\mathbb{Q})),$$

as follows:

$$P_{ij} := p_{13*}\left(p_{124}^*\left[Q^{[n,\,n-i]}\right]\cdot p_{235}^*\left[Q^{[n-i+j,\,n-i]}\right]\right),$$

$$\Big(\text{resp.}\qquad Q_{ij} := p_{13*}\left(p_{124}^*\left[Q^{[n+j,\,n]}\right]\cdot p_{235}^*\left[Q^{[n+j,\,n-i+j]}\right]\right)\Big),$$

where $Q^{[r,\,s]}$ is defined in (2). Then $\mathfrak{q}_j(\beta)\mathfrak{q}_{-i}(\alpha)$, (resp. $\mathfrak{q}_{-i}(\alpha)\mathfrak{q}_j(\beta)$), is given by

$$\tau \longmapsto PD^{-1}\left[\mathrm{pr}_{3*}\left(P_{ij}\cap\left(\mathrm{pr}_5^*\,\beta\cup\mathrm{pr}_4^*\,\alpha\cup\mathrm{pr}_1^*\,\tau\right)\right)\right],\qquad \Big(\text{resp.}$$

$$\tau \longmapsto PD^{-1}\left[\mathrm{pr}_{3*}\left(Q_{ij}\cap\left(\mathrm{pr}_5^*\,\alpha\cup\mathrm{pr}_4^*\,\beta\cup\mathrm{pr}_1^*\,\tau\right)\right)\right]\Big).$$

First we deform all the relative integrable complex structures into a single one parameterized by $X^{(n)}\times X^{(n-i)}\times X^{(n-i+j)}\times X^{(2)}$.

Let $J^{\text{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ be a relative integrable complex structure in a neighbourhood of $Z_{n\times(n-i)\times(n-i+j)\times 2}$, and let us define $J^{\text{rel}}_{n\times(n-i)\times(n-i+j)\times 1\times 1}$ by the formula $J^{\text{rel}}_{n\times(n-i)\times(n-i+j)\times 1\times 1,\,\underline{x},\,\underline{y},\,\underline{z},\,s,\,t} = J^{\text{rel}}_{n\times(n-i)\times(n-i+j)\times 2,\,\underline{x},\,\underline{y},\,\underline{z},\,s\cup t}$.

If $Y=\left(X^{[n]\times[n-i]\times[n-i+j]\times[1]\times[1]},\,J^{\text{rel}}_{n\times(n-i)\times(n-i+j)\times 1\times 1}\leftarrow 5\text{ times}\right)$ is the product Hilbert scheme obtained by taking the same relative integrable complex structure $J^{\text{rel}}_{n\times(n-i)\times(n-i+j)\times 1\times 1}$ five times (see Definition 4), there is a canonical isomorphism between $H_*(Y,\mathbb{Q})$ and

$$H_*(X^{[n]},\mathbb{Q})\otimes H_*\left(X^{[n-i]},\mathbb{Q}\right)\otimes H_*\left(X^{[n-i+j]},\mathbb{Q}\right)\otimes H_*(X,\mathbb{Q})\otimes H_*(X,\mathbb{Q})\,.$$

Since incidence varieties vary trivially in families, the class $p_{124}^*\left[Q^{[n,n-i]}\right]$ is the homology class of the cycle

$$A=\left\{(\xi,\xi',\xi'',s,t)\text{ in }Y\text{ such that }\xi'\subseteq\xi\text{ and }\rho(\xi',\xi)=s\right\}\cdot$$

In the same way, $p_{235}^*\left[Q^{[n-i+j,n-i]}\right]$ is the homology class of the cycle

$$B=\left\{(\xi,\xi',\xi'',s,t)\text{ in }Y\text{ such that }\xi'\subseteq\xi''\text{ and }\rho(\xi',\xi'')=t\right\}\cdot$$

Let us study the intersection of the cycles A and B: choose relative holomorphic coordinates $\phi_{\underline{x},\underline{y},\underline{z},s,t}$ for $J^{\text{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ such that the map

$$(\underline{x},\underline{y},\underline{z},s,t)\longmapsto\Big(\phi^{(n)}_{\underline{x},\,\underline{y},\,\underline{z},\,s,\,t}(\underline{x}),\,\phi^{(n-i)}_{\underline{x},\,\underline{y},\,\underline{z},\,s,\,t}(\underline{y}),\,\phi^{(n-i+j)}_{\underline{x},\,\underline{y},\,\underline{z},\,s,\,t}(\underline{z}),$$
$$\phi_{\underline{x},\,\underline{y},\,\underline{z},\,s,\,t}(s),\,\phi_{\underline{x},\,\underline{y},\,\underline{z},\,s,\,t}(t)\Big)$$

defined in an open subset V of $X^{(n)}\times X^{(n-i)}\times X^{(n-i+j)}\times X\times X$ with values in $(\mathbb{C}^2)^{(n)}\times(\mathbb{C}^2)^{(n-i)}\times(\mathbb{C}^2)^{(n-i+j)}\times\mathbb{C}^2\times\mathbb{C}^2$ is a homeomorphism between V and its image, and is equivariant for the $\mathbb{Z}/2\mathbb{Z}$–action which permutes the two last factors in the domain and in the range. If p is a point in $A \cap B$ lying over V, the map given by

$$(\xi, \underline{x}, \xi', \underline{y}, \xi'', \underline{z}, s, t) \longmapsto \Big(\phi^{(n)}_{\underline{x}, \underline{y}, \underline{z}, s, t\,*}\xi, \phi^{(n-i)}_{\underline{x}, \underline{y}, \underline{z}, s, t\,*}\xi', \phi^{(n-i+j)}_{\underline{x}, \underline{y}, \underline{z}, s, t*}\xi'',$$

$$\phi_{\underline{x}, \underline{y}, \underline{z}, s, t}(s), \phi_{\underline{x}, \underline{y}, \underline{z}, s, t}(t)\Big)$$

is a homeomorphism from a neighbourhood of p in the product Hilbert scheme Y to its image in $(\mathbb{C}^2)^{[n]} \times (\mathbb{C}^2)^{[n-i]} \times (\mathbb{C}^2)^{[n-i+j]} \times \mathbb{C}^2 \times \mathbb{C}^2$ and maps A and B to the classical cycles $p_{124}^{-1} Q^{[n,n-i]}$ and $p_{235}^{-1} Q^{[n-i+j,n-i]}$. In the integrable case, we know that in the open set $\{s \neq t\}$, $p_{124}^{-1} Q^{[n,n-i]}$ and $p_{235}^{-1} Q^{[n-i+j,n-i]}$ intersect generically transversally. Thus, this property still holds in our context. If $(A \cap B)_{s \neq t} = C_{ij}$ we can write $[A] \cdot [B] = \Big[\overline{C_{ij}}\Big] + \iota_* R$, where $\iota \colon Y_{\{s=t\}} \hookrightarrow Y$ is the natural injection and R is in $H_{2(2n-i+j+2)}\Big(Y_{\{s=t\}}, \mathbb{Q}\Big)$.

We can proceed similarly in the product Hilbert scheme

$$Y' = \Big(X^{[n]\times[n+j]\times[n-i+j]\times[1]\times[1]}, J^{\text{rel}}_{n\times(n+j)\times(n-i+j)\times 1\times 1} \leftarrow 5 \text{ times}\Big)$$

with the cycles A' and B' defined by

$$A' = \big\{(\xi, \xi', \xi'', s, t) \text{ in } Y' \text{ such that } \xi \subseteq \xi' \text{ and } \rho(\xi, \xi') = s\big\}$$
$$B' = \big\{(\xi, \xi', \xi'', s, t) \text{ in } Y' \text{ such that } \xi'' \subseteq \xi' \text{ and } \rho(\xi'', \xi') = t\big\}\cdot$$

Let $D_{ij} = \big(A' \cap B'\big)_{s \neq t}$. Then, we can write $[A'] \cdot [B'] = \Big[\overline{D_{ij}}\Big] + \iota'_* R'$, where $\iota' \colon Y'_{\{s=t\}} \hookrightarrow Y'$ is the injection and R' is in $H_{2(2n-i+j+2)}\Big(Y'_{\{s=t\}}, \mathbb{Q}\Big)$. The homology class R (resp. R') can be chosen supported in $A \cap B \cap Y_{\{s=t\}}$ (resp. in $A' \cap B' \cap Y'_{\{s=t\}}$).

The following lemma describes the situation outside the diagonal $\{s = t\}$:

Lemma 2.

$$p_{1345*}\Big(\Big[\overline{C_{ij}}\Big] \cap \big(\text{pr}_5^* \beta \cup \text{pr}_4^* \alpha\big)\Big) = (-1)^{|\alpha|\,|\beta|} p_{1345*}\Big(\Big[\overline{D_{ij}}\Big] \cap \big(\text{pr}_5^* \alpha \cup \text{pr}_4^* \beta\big)\Big).$$

Proof. We introduce the incidence varieties

$$T = \Big\{(\underline{x}, \underline{y}, \underline{z}, s, t) \text{ in } X^{(n)} \times X^{(n-i)} \times X^{(n-i+j)} \times X \times X \text{ such that } \underline{x} = \underline{y} \cup is \text{ and } \underline{z} = \underline{y} \cup jt\Big\}$$

$$T' = \Big\{(\underline{x}, \underline{y}, \underline{z}, s, t) \text{ in } X^{(n)} \times X^{(n+j)} \times X^{(n-i+j)} \times X \times X \text{ such that } \underline{y} = \underline{x} \cup js = \underline{z} \cup it\Big\}\cdot$$

We choose two small neighbourhoods Ω, Ω' of T and T' and a neighbourhood W of $Z_{n\times(n-i+j)\times 2}$ such that for any $(\underline{x}, \underline{y}, \underline{z}, s, t)$ in Ω (resp. Ω'), $\underline{y}$ is in $W_{\underline{x},\underline{z},s\cup t}$. Let $J^{\text{rel}}_{n\times(n-i+j)\times 2}$ be a relative integrable complex structure on W. After shrinking

Ω and Ω' if necessary, we can consider two relative structures $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ and $J^{\mathrm{rel}}_{n\times(n+j)\times(n-i+j)\times 2}$ such that

$$\begin{cases} \forall(\underline{x},\underline{y},\underline{z},s,t)\in\Omega, & J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2,\underline{x},\underline{y},\underline{z},s\,\cup\, t} = J^{\mathrm{rel}}_{n\times(n-i+j)\times 2,\underline{x},\underline{z},s\,\cup\, t} \\ \forall(\underline{x},\underline{y},\underline{z},s,t)\in\Omega', & J^{\mathrm{rel}}_{n\times(n+j)\times(n-i+j)\times 2,\underline{x},\underline{y},\underline{z},s\,\cup\, t} = J^{\mathrm{rel}}_{n\times(n-i+j)\times 2,\underline{x},\underline{z},s\,\cup\, t} \end{cases}$$

Let U (resp. U') be the points of Y (resp. Y') lying over Ω (resp. Ω'). We define a relative integrable complex structure $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 1\times 1}$ by the formula

$$J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 1\times 1,\,\underline{x},\,\underline{y},\,\underline{z},\,s,\,t} = J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2,\,\underline{x},\,\underline{y},\,\underline{z},\,s\,\cup\, t},$$

as well as two maps u and v:

$$\begin{aligned} u\colon\ U &\longrightarrow \left(X^{[n]\times[n-i+j]\times[1]\times[1]},\, J^{\mathrm{rel}}_{n\times(n-i+j)\times 1\times 1} \leftarrow 4 \text{ times}\right) \\ (\xi,\xi',\xi'',s,t) &\longmapsto\ (\xi,\xi'',s,t), \end{aligned}$$

$$\begin{aligned} v\colon\ U' &\longrightarrow \left(X^{[n]\times[n-i+j]\times[1]\times[1]},\, J^{\mathrm{rel}}_{n\times(n-i+j)\times 1\times 1} \leftarrow 4 \text{ times}\right), \\ (\xi,\xi',\xi'',s,t) &\longmapsto\ (\xi,\xi'',s,t) \end{aligned}$$

If we take three homeomorphisms

$$\begin{cases} X^{[n]}\times X^{[n-i]}\times X^{[n-i+j]}\times X^2 \simeq X^{[n]\times[n-i]\times[n-i+j]\times[1]\times[1]} \\ X^{[n]}\times X^{[n+j]}\times X^{[n-i+j]}\times X^2 \simeq X^{[n]}\times X^{[n-i+j]}\times X^2 \\ X^{[n]}\times X^{[n-i+j]}\times X^2 \simeq X^{[n]\times[n-i+j]\times[1]\times[1]} \end{cases}$$

then u and v can be extended to global maps which are in the homotopy class of p_{1345}.

As in the integrable case, there is a natural isomorphism $\phi : C_{ij} \xrightarrow{\ \simeq\ } D_{ij}$ described as follows: if (ξ,ξ',ξ'',s,t) is an element of C_{ij} with

$$\Gamma(\xi') = \underline{y},\ \Gamma(\xi) = \underline{y}\cup is \ \text{ and } \ \Gamma(\xi'') = \underline{y}\cup jt,$$

then $\phi(\xi,\xi',\xi'',s,t) = (\xi,\widetilde{\xi},\xi'',t,s)$ where $\widetilde{\xi}$ is defined by

$$\begin{cases} \widetilde{\xi}_{|\,p} = \xi'_{|\,p} & \text{for } p\in\underline{y},\ \ p\notin\{s,t\} \\ \widetilde{\xi}_{|\,s} = \xi_{|\,s} & \text{and } \widetilde{\xi}_{|\,t} = \xi''_{|\,t}, \end{cases}$$

these schemes being considered for the structure $J^{\mathrm{rel}}_{n\times(n-i+j)\times 2,\underline{y}\,\cup\, is,\underline{y}\,\cup\, jt,s\,\cup\, t}$.

Let $\partial C_{ij} = \overline{C_{ij}}\setminus C_{ij}$, $\partial D_{ij} = \overline{D_{ij}}\setminus D_{ij}$ and $S = u(\partial C_{ij}) = v(\partial D_{ij})$; we define a map $\pi :\ Y' \longrightarrow\ Y'$ by $\pi(\xi,\xi',\xi'',s,t) = (\xi,\xi',\xi'',t,s)$. Then we have the following diagram, where all the maps are proper:

$$\begin{array}{ccccc} Y\setminus\partial C_{ij} \supseteq C_{ij} & & \xrightarrow[\ \simeq\ ]{\ \phi\ } & & D_{ij} \subseteq Y'\setminus\partial D_{ij}\,. \\ & \searrow^{u} & & \swarrow^{v\circ\pi} & \\ & & X^{[n]\times[n-i+j]\times[1]\times[1]}\setminus S & & \end{array}$$

Thus we obtain $u_*\Big([C_{ij}]\cap(\mathrm{pr}_5^*\,\beta\cup\mathrm{pr}_4^*\,\alpha)\Big) = v_*\Big([D_{ij}]\cap(\mathrm{pr}_4^*\,\beta\cup\mathrm{pr}_5^*\,\alpha)\Big)$ in the Borel-Moore homology group $H^{\mathrm{lf}}_{2(2n-i+j+2)}\Big(X^{[n]\times[n-i+j]\times[1]\times[1]}\backslash S,\mathbb{Q}\Big)$.

Since $\dim S\leq 2(2n-i+j+2)-2$, we get

$$p_{1345*}\Big(\Big[\,\overline{C_{ij}}\,\Big]\cap\big(\mathrm{pr}_5^*\,\beta\cup\mathrm{pr}_4^*\,\alpha\big)\Big)=(-1)^{|\alpha|\,|\beta|}p_{1345*}\Big(\Big[\,\overline{D_{ij}}\,\Big]\cap\big(\mathrm{pr}_5^*\,\alpha\cup\mathrm{pr}_4^*\,\beta\big)\Big).$$

□

By this lemma, the terms coming from $\overline{C_{ij}}$ and $\overline{D_{ij}}$ in $\big[\mathfrak{q}_{-i}(\alpha),\mathfrak{q}_j(\beta)\big]$ cancel out. It remains to deal with the excess intersection components along the diagonals $Y_{\{s=t\}}$ and $Y'_{\{s=t\}}$. We introduce the locus

$$Z=\Big\{\Big(\xi,\ \underline{x},\ \xi'',\underline{z},\ s,\ t\Big)\text{ in }X^{[n]\times[n-i+j]\times[1]\times[1]}\text{ such that }s=t,\ \xi_{\,|\,p}=\xi''_{\,|\,p}$$
$$\text{for }p\neq s,\ \Gamma(\xi'')=\Gamma(\xi)+(j-i)s\text{ if }j\geq i\quad\text{and}$$
$$\Gamma(\xi)=\Gamma(\xi'')+(i-j)s\text{ if }j\leq i\Big\}.$$

Then Z contains $u(A\cap B)$ and $v(A'\cap B')$. As before, the dimension count can be done as in the integrable case: if $i\neq j$, $\dim Z<2(2n-i+j+2)$ and if $i=j$, Z contains a $2(2n+2)$–dimensional component, namely $\Delta_{X^{[n]}}\times\Delta_X$. All other components have lower dimensions. Thus, if $i\neq j$, $p_{1345*}(\iota_*R)$ and $p_{1345*}(\iota'_*R')$ vanish since these two homology classes are supported in Z and their degree is $2(2n-i+j+2)$. In the case $i=j$, then $p_{1345*}(\iota_*R)$ and $p_{1345*}(\iota'_*R')$ are proportional to the fundamental homology class of $\Delta_{X^{[n]}}\times\Delta_X$. Now $p_{45*}\Big(\Big[\Delta_{X^{[n]}}\times\Delta_X\Big]\cap\big(\mathrm{pr}_4^*\,\alpha\cup\mathrm{pr}_5^*\,\beta\big)\Big)=\int_X\alpha\beta.\Big[\Delta_{X^{[n]}}\Big]$ and we obtain the identity $\Big[\mathfrak{q}_{-i}(\alpha),\mathfrak{q}_i(\beta)\Big]=\mu\int_X\alpha\beta.\,\mathrm{id}$ where μ is a rational number. The computation of the multiplicity μ is a local problem on X which is solved in [6,13]; it turns out that $\mu=-i$. □

Remark 4. The proof remains quite similar for $i>0$, $j>0$. There is no excess term in this case. Indeed,

$$Y=X^{[n]\times[n+i]\times[n+i+j]\times[1]\times[1]},\ \ Z=X^{[n+i+j,n]}\subseteq X^{[n]\times[n+i]\times[n+i+j]\times[1]\times[1]}$$

and $\dim Z=2(2n+i+j+1)<2(2n+i+j+2)$.

Theorem 6 gives a representation in $\mathbb{H}=\bigoplus_{n\in\mathbb{N}}H^*(X^{[n]},\mathbb{Q})$ of the Heisenberg super-algebra $\mathcal{H}(H^*(X,\mathbb{Q}))$ of $H^*(X,\mathbb{Q})$.

Proposition 4. *$\mathbb{H}$ is an irreducible $\mathcal{H}(H^*(X,\mathbb{Q}))$–module with highest weight vector 1.*

This a consequence of Theorem 6 and Göttsche's formula (Theorem 5), as shown by Nakajima [17].

5. Tautological bundles

5.1. Construction of the tautological bundles

Our aim in this section is to associate to any complex vector bundle E on an almost-complex compact four-manifold X a collection of complex vector bundles $E^{[n]}$ on $X^{[n]}$ which generalize the tautological bundles already known in the algebraic context.

Let (X, J) be an almost-complex compact four-manifold, $Z_n \subseteq X^{(n)} \times X$ the incidence locus, W be a small neighbourhood of Z_n in $X^{(n)} \times X$ and J_n^{rel} be a relative integrable structure on W. The fibers of $\text{pr}_1 : W \longrightarrow X^{(n)}$ are smooth analytic sets. We endow W with the sheaf $\mathcal{A}_W$ of continuous functions which are smooth on the fibers of pr_1. We can consider the sheaf $\mathcal{A}^{0,1}_{W,\text{rel}}$ of relative $(0, 1)$–forms on W. There exists a relative $\overline{\partial}$–operator $\overline{\partial}^{\text{rel}} : \mathcal{A}_W \longrightarrow \mathcal{A}^{0,1}_{W,\text{rel}}$ which induces for each $\underline{x} \in X^{(n)}$ the usual operator $\overline{\partial} : \mathcal{A}_{W_{\underline{x}}} \longrightarrow \mathcal{A}^{0,1}_{W_{\underline{x}}}$ given by the complex structure $J^{\text{rel}}_{n,\underline{x}}$ on $W_{\underline{x}}$.

Definition 7. Let E be a complex vector bundle on X.

(i) A *relative connection* $\overline{\partial}^{\text{rel}}_E$ *on E compatible with* J_n^{rel} is a $\mathbb{C}$–linear morphism of sheaves $\overline{\partial}_E : \mathcal{A}_W\left(\text{pr}_2^* E\right) \longrightarrow \mathcal{A}^{0,1}_W\left(\text{pr}_2^* E\right)$ satisfying a relative Leibniz rule: $\overline{\partial}^{\text{rel}}_E(\varphi s) = \varphi\, \overline{\partial}^{\text{rel}}_E s + \overline{\partial}^{\text{rel}} \varphi \otimes s$ for all sections φ and s of $\mathcal{A}_W\left(\text{pr}_2^* E\right)$ and $\mathcal{A}_W$ respectively.

(ii) A relative connection $\overline{\partial}^{\text{rel}}_E$ on E is *integrable* if $\left(\overline{\partial}^{\text{rel}}_E\right)^2 = 0$.

If $\overline{\partial}^{\text{rel}}_E$ is an integrable connection on E compatible with J_n^{rel}, we can apply the Koszul-Malgrange integrability theorem with continuous parameters in $X^{(n)}$ (see [22]). Thus, for every $\underline{x} \in X^{(n)}$, $E_{|W_{\underline{x}}}$ is endowed with the structure of a holomorphic vector bundle over $\left(W_{\underline{x}}, J^{\text{rel}}_{n,\underline{x}}\right)$ and this structure varies continuously with $\underline{x}$. Furthermore, $\ker \overline{\partial}^{\text{rel}}_E$ is the sheaf of relative holomorphic sections of E. Therefore, there is no difference between relative integrable connections on E compatible with J_n^{rel} and relative holomorphic structures on E compatible with J_n^{rel}.

Taking relative holomorphic coordinates for J_n^{rel}, we can see that relative integrable connections exist on W over small open sets of $X^{(n)}$. By a partition of unity on $X^{(n)}$, it is possible to build global ones. Besides, the space of holomorphic structures on a complex vector bundle over a ball in $\mathbb{C}^2$ is contractible, so that the space of relative holomorphic structures on E compatible with J_n^{rel} is also contractible.

We proceed now to the construction of the tautological bundle $E^{[n]}$ on $X^{[n]}_{J_n^{\text{rel}}}$. Let $\overline{\partial}^{\text{rel}}_E$ be a relative holomorphic structure on E adapted to J_n^{rel}. Taking relative holomorphic coordinates, we get a vector bundle $E^{[n]}_{\text{rel}}$ over $W^{[n]}_{\text{rel}}$ satisfying the following property: for each $\underline{x}$ in $X^{(n)}$, $E^{[n]}_{\text{rel}\,|W^{[n]}_{\underline{x}}} = E^{[n]}_{|W_{\underline{x}}}$, where $E_{|W_{\underline{x}}}$ is endowed with the holomorphic structure $\overline{\partial}^{\text{rel}}_{E,\underline{x}}$.

Definition 8. Let $i : X^{[n]}_{J_n^{\text{rel}}} \longrightarrow W^{[n]}_{\text{rel}}$ be the canonical injection. The complex vector bundle $\left(E^{[n]}, J_n^{\text{rel}}, \overline{\partial}^{\text{rel}}_E\right)$ on $X^{[n]}_{J_n^{\text{rel}}}$ is defined by $E^{[n]} = i^* E^{[n]}_{\text{rel}}$.

In the sequel, we consider the class of $E^{[n]}$ in $K(X^{[n]})$, which we prove below to be independent of the structures used in the construction.

Proposition 5. *The class of* $E^{[n]}$ *in* $K(X^{[n]})$ *is independent of* $\left(J_n^{\mathrm{rel}}, \overline{\partial}_E^{\mathrm{rel}}\right)$.

Proof. Let $\left(J_{0,n}^{\mathrm{rel}}, \overline{\partial}_{E,0}^{\mathrm{rel}}\right)$ and $\left(J_{1,n}^{\mathrm{rel}}, \overline{\partial}_{E,1}^{\mathrm{rel}}\right)$ be two relative holomorphic structures on E, $\left(J_{t,n}^{\mathrm{rel}}, \overline{\partial}_{E,t}^{\mathrm{rel}}\right)$ be a smooth path between them, and $W_{\mathrm{rel}}^{[n]}$ be the relative Hilbert scheme over $X^{(n)} \times [0, 1]$ for the family $\left(J_{t,n}^{\mathrm{rel}}\right)_{0 \le t \le 1}$. There exists a vector bundle $\left(\widetilde{E}_{\mathrm{rel}}^{[n]}, \left\{J_{t,n}^{\mathrm{rel}}\right\}_{0 \le t \le 1}, \left\{\overline{\partial}_{E,t}^{\mathrm{rel}}\right\}_{0 \le t \le 1}\right)$ over $W_{\mathrm{rel}}^{[n]}$ such that for all t in $[0, 1]$, $\widetilde{E}^{[n]}_{\mathrm{rel}\,|\,W_{\mathrm{rel},t}^{[n]}} = \left(E_{\mathrm{rel}}^{[n]}, J_{t,n}^{\mathrm{rel}}, \overline{\partial}_{E,t}^{\mathrm{rel}}\right)$. If $\mathfrak{X} = \left(X^{[n]}, \left\{J_{t,n}^{\mathrm{rel}}\right\}_{0 \le t \le 1}\right) \subseteq W_{\mathrm{rel}}^{[n]}$ is the relative Hilbert scheme over $[0, 1]$, then $\widetilde{E}^{[n]}_{\mathrm{rel}\,|\,\mathfrak{X}}$ is a complex vector bundle on $\mathfrak{X}$ whose restriction to $\mathfrak{X}_t$ is $\left(E^{[n]}, J_{t,n}^{\mathrm{rel}}, \overline{\partial}_{E,t}^{\mathrm{rel}}\right)$. Now $\mathfrak{X}$ is topologically trivial over $[0, 1]$ by Proposition 2. Since $K(\mathfrak{X}_0 \times [0, 1]) \simeq K(\mathfrak{X}_0)$, we get the result. □

Let us give an important example. If $\mathbb{T} = X \times \mathbb{C}$ is the trivial complex line bundle on X, the tautological bundles $\mathbb{T}^{[n]}$ already convey geometric information on $X^{[n]}$. To see this, let $\partial X^{[n]} \subseteq X^{[n]}$ be the inverse image of the big diagonal of $X^{(n)}$ by the Hilbert–Chow morphism. We have $\dim \partial X^{[n]} = 4n - 2$ and $H_{4n-2}(\partial X^{[n]}, \mathbb{Z}) \simeq \mathbb{Z}$ (this can be proved as in Lemma 1).

Lemma 3. $c_1(\mathbb{T}^{[n]}) = -\frac{1}{2}\, PD^{-1}([\partial X^{[n]}])$ *in* $H^2(X^{[n]}, \mathbb{Q})$.

Proof. Let $U = \left\{(x_1, \ldots, x_n) \text{ in } X^{[n]} \text{ such that for } i \neq j,\ x_i \neq x_j\right\}$. Then $X^{[n]} \setminus \partial X^{[n]}$ is canonically isomorphic to $U/\mathfrak{S}_n$. If $\sigma : U \longrightarrow X^{[n]} \setminus \partial X^{[n]}$ is the associated quotient map, $\sigma^*\mathbb{T}^{[n]} \simeq \bigoplus_{i=1}^n \mathrm{pr}_i^*\, \mathbb{T}$, so that $\sigma^*\mathbb{T}^{[n]}$ is trivial. Since σ is a finite covering map, $c_1(\mathbb{T}^{[n]})_{|X^{[n]} \setminus \partial X^{[n]}}$ is a torsion class, so it is zero in $H^2(X^{[n]} \setminus \partial X^{[n]}, \mathbb{Q})$. This implies that $c_1(\mathbb{T}^{[n]})$ is Poincaré dual to a rational multiple of $[\partial X^{[n]}]$. To compute the proportionality coefficient μ, we argue locally on $X^{(n)}$ around a point in the stratum

$$S = \left\{\underline{x} \in X^{(n)} \text{ such that } x_i \neq x_j \text{ except for one pair } \{i, j\}\right\} \cdot$$

This reduces the computation to the case $n = 2$. For any open subset U of X endowed with an integrable complex structure, if Δ is the diagonal of U, then $U^{[2]} = Bl_\Delta(U \times U)/\mathbb{Z}_2$. Besides, if $E \subseteq Bl_\Delta(U \times U)$ is the exceptional divisor and π is the projection from $Bl_\Delta(U \times U)/\mathbb{Z}_2$ to $U^{[2]}$, then $\pi^*\left([\partial U^{[2]}]\right) = 2[E]$. Thus we obtain:

$$\pi^* c_1(\mathbb{T}^{[2]}) = c_1(\pi^*\mathbb{T}^{[2]}) = c_1(\mathcal{O}(-E)) = -[E] \quad \text{in} \quad H^2\left(Bl_\Delta(U \times U), \mathbb{Z}\right).$$

This gives the value $\mu = -1/2$. □

5.2. Tautological bundles and incidence varieties

We want to compare the tautological bundles $E^{[n]}$ and $E^{[n+1]}$ through the incidence variety $X^{[n+1,n]}$. In the integrable case, $X^{[n+1,n]}$ is smooth. If D denotes the divisor $\overline{Z}_1$ in $X^{[n+1,n]}$ (see (1)), we have an exact sequence (see [4,16]):

$$0 \longrightarrow \rho^* E \otimes \mathcal{O}_{X^{[n+1,n]}}(-D) \longrightarrow \nu^* E^{[n+1]} \longrightarrow \lambda^* E^{[n]} \longrightarrow 0, \qquad (3)$$

where $\lambda\colon\ X^{[n+1,n]} \longrightarrow X^{[n]}$; $\nu\colon\ X^{[n+1,n]} \longrightarrow X^{[n+1]}$ and $\rho\colon\ X^{[n+1,n]} \longrightarrow X$ are the two natural projections and the residual map.

In the almost-complex case, $X^{[n+1,n]}$ is a topological manifold of dimension $4n+4$ (if we choose a relative integrable structure J_{n+1}^{rel} with additional properties as given in [20], $X^{[n+1,\,n]}$ can be endowed with a differentiable structure, but we will not need it here).

Let J_n^{rel} and J_{n+1}^{rel} be two relative integrable structures in small neighbourhoods of Z_n and Z_{n+1}. We extend them to relative integrable complex structures $\check{J}^{1,\,\mathrm{rel}}_{n\times(n+1)}$ and $\check{J}^{2,\,\mathrm{rel}}_{n\times(n+1)}$ in small neighbourhoods of the incidence set $Z_{n\times(n+1)}$. Then $\left(X^{[n]\times[n+1]}, \check{J}^{1,\,\mathrm{rel}}_{n\times(n+1)}, \check{J}^{2,\,\mathrm{rel}}_{n\times(n+1)}\right) = X^{[n]}_{J_n^{\mathrm{rel}}} \times X^{[n+1]}_{J_{n+1}^{\mathrm{rel}}}$. If $J^{\mathrm{rel}}_{n\times(n+1)}$ is a relative integrable structure in a small neighbourhood of $Z_{n\times(n+1)}$ and $J^{\mathrm{rel}}_{n\times 1}$ is defined for all $(\underline{x},\, p)$ in $X^{(n)}\times X$ by $J^{\mathrm{rel}}_{n\times 1,\,\underline{x},\,p} = J^{\mathrm{rel}}_{n\times(n+1),\,\underline{x},\,\underline{x}\cup p}$, then we have a commutative diagram in the homotopy category:

$$\begin{array}{ccccc}
 & & X^{[n+1,n]}_{J^{\mathrm{rel}}_{n\times 1}} & & \\
 & \swarrow^{\lambda} & \big\downarrow & {}^{\nu}\searrow & \\
 & & \left(X^{[n]\times[n+1]}, J^{\mathrm{rel}}_{n\times(n+1)}, J^{\mathrm{rel}}_{n\times(n+1)}\right) & & \\
 & & \simeq \big\downarrow \Phi & & \\
X^{[n]}_{J_n^{\mathrm{rel}}} & \xleftarrow[\mathrm{pr}_2]{} & \left(X^{[n]\times[n+1]}, \check{J}^{1,\,\mathrm{rel}}_{n\times(n+1)}, \check{J}^{2,\,\mathrm{rel}}_{n\times(n+1)}\right) & \xrightarrow[\mathrm{pr}_1]{} & X^{[n+1]}_{J^{\mathrm{rel}}_{n+1}}
\end{array}$$

where Φ is a homeomorphism whose homotopy class is canonical. Let us denote by D the inverse image of the incidence locus Z_n in $X^{(n)}\times X$ by the map $X^{[n+1,n]} \longmapsto X^{(n)}\times X$, so that $D = \overline{Z}_1$ where Z_1 is defined by (1). The cycle D has a fundamental homology class in $H_{4n+2}\left(X^{[n+1,n]}, \mathbb{Z}\right)$, and this last homology group is in fact isomorphic to $\mathbb{Z}$ (see the proof of Lemma 1). Furthermore, there exists a unique complex line bundle F on $X^{[n+1,n]}$ such that $PD^{-1}\left(c_1(F)\right) = -[D]$.

Proposition 6. *The identity* $\nu^* E^{[n+1]} = \lambda^* E^{[n]} + \rho^* E \otimes F$ *holds in* $K\left(X^{[n+1,n]}\right)$.

Proof. Let $\overline{\partial}^{\,\text{rel}}_{E,n\times 1}$, $\overline{\partial}^{\,\text{rel}}_{E,n}$ and $\overline{\partial}^{\,\text{rel}}_{E,n+1}$ be relative holomorphic structures on E compatible with $J^{\text{rel}}_{n\times 1}$, J^{rel}_{n} and J^{rel}_{n+1}. For each $(\underline{x}, p)$ in $X^{(n)} \times X$, we consider the exact sequence (3) on $\left(W_{\underline{x},p}, J^{\text{rel}}_{n\times 1,\underline{x},p}\right)$ for the holomorphic vector bundle $\left(E_{|W_{\underline{x},p}}, \overline{\partial}^{\,\text{rel}}_{E,n\times 1,\underline{x},p}\right)$. Putting these exact sequences in family over $X^{(n)} \times X$ and taking the restriction to $X^{[n+1,n]}$, we get an exact sequence

$$0 \longrightarrow \rho^* E \otimes G \longrightarrow A \longrightarrow B \longrightarrow 0,$$

where G is a complex line bundle on $X^{[n+1,n]}$ and A and B are two vector bundles on $X^{[n+1,n]}$ such that for all $(\underline{x}, p)$ in $X^{(n)} \times X$

$$\begin{cases} A_{|\xi,\xi',\underline{x},p} = \left(E^{[n+1]}_{|\xi'}, \overline{\partial}^{\,\text{rel}}_{E,n\times 1,\underline{x},p}, J^{\text{rel}}_{n\times 1,\underline{x},p}\right) \\ B_{|\xi,\xi',\underline{x},p} = \left(E^{[n]}_{|\xi}, \overline{\partial}^{\,\text{rel}}_{E,n\times 1,\underline{x},p}, J^{\text{rel}}_{n\times 1,\underline{x},p}\right) \end{cases} \tag{4}$$

Let us write $\Phi(\xi, \xi', \underline{u}, \underline{v}) = \left(\phi_{\underline{u},\underline{v}*}\xi,\ \psi_{\underline{u},\underline{v}*}\xi',\ \phi^{(n)}_{\underline{u},\underline{v}}(\underline{u}),\ \psi^{(n+1)}_{\underline{u},\underline{v}*}(\underline{v})\right)$ where, for $(\underline{u}, \underline{v})$ in $X^{(n)} \times X^{(n+1)}$, the map $\phi_{\underline{u},\underline{v}}$ (resp. $\psi_{\underline{u},\underline{v}}$) is a biholomorphism between a neighbourhood of supp $(\underline{u} \cup \underline{v})$ endowed with the complex structure $J^{\text{rel}}_{n\times(n+1),\underline{u},\underline{v}}$ and its image in X endowed with the structure $\check{J}^{\,1,\text{rel}}_{n\times(n+1),\phi^{(n)}_{\underline{u},\underline{v}}(\underline{u}),\phi^{(n+1)}_{\underline{u},\underline{v}}(\underline{v})}$ (resp. $\check{J}^{\,2,\text{rel}}_{n\times(n+1),\psi^{(n)}_{\underline{u},\underline{v}}(\underline{u}),\psi^{(n+1)}_{\underline{u},\underline{v}}(\underline{v})}$). Then

$$\nu^* E^{[n+1]}_{|\xi,\xi',\underline{x},p} = \left(E^{[n+1]}_{|\psi_{\underline{x},\underline{x}\cup p*}\xi'}, \overline{\partial}^{\,\text{rel}}_{E,n+1,\psi^{(n+1)}_{\underline{x},\underline{x}\cup p}(\underline{x}\cup p)}, J^{\text{rel}}_{n+1,\psi^{(n+1)}_{\underline{x},\underline{x}\cup p}(\underline{x}\cup p)}\right),$$

$$\lambda^* E^{[n]}_{|\xi,\xi',\underline{x},p} = \left(E^{[n]}_{|\phi_{\underline{x},\underline{x}\cup p*}\xi}, \overline{\partial}^{\,\text{rel}}_{E,n,\phi^{(n)}_{\underline{x},\underline{x}\cup p}(\underline{x})}, J^{\text{rel}}_{n,\phi^{(n)}_{\underline{x},\underline{x}\cup p}(\underline{x})}\right).$$

As in Proposition 5, the classes A and B in $K\left(X^{[n+1,n]}\right)$ are independent of the relative holomorphic structures used to define them.

- If $J^{\text{rel}}_{n\times(n+1)} = \check{J}^{\,2,\text{rel}}_{n\times(n+1)}$ and if for all $(\underline{x}, p)$ in $X^{(n)} \times X$, $\overline{\partial}^{\,\text{rel}}_{E,n\times 1,\underline{x},p}$ and $\overline{\partial}^{\,\text{rel}}_{E,n+1,\underline{x}\cup p}$ are equal, we can assume that $\psi_{\underline{u},\underline{v}} = \text{id}$ in a neighbourhood of supp$(\underline{v})$. Thus $A = \nu^* E^{[n+1]}$.
- On the other way, if $J^{\text{rel}}_{n\times(n+1)} = \check{J}^{\,1,\text{rel}}_{n\times(n+1)}$ and for all $(\underline{x}, p)$ in $X^{(n)} \times X$, $\overline{\partial}^{\,\text{rel}}_{E,n\times 1,\underline{x},p} = \overline{\partial}^{\,\text{rel}}_{E,n,\underline{x}}$ in a neighbourhood of supp$(\underline{x})$, we can take $\phi_{\underline{u},\underline{v}} = \text{id}$ in a neighbourhood of supp$(\underline{u})$. Thus $B = \lambda^* E^{[n]}$.

This proves that $\nu^*E^{[n+1]} - \lambda^*E^{[n]} = \rho^*E \otimes G$ in $K\left(X^{[n+1,n]}\right)$. If $\mathbb{T}$ is the trivial complex line bundle on X, $\nu^*\mathbb{T}^{[n+1]} \simeq \lambda^*\mathbb{T}^{[n]} \oplus \rho^*\mathbb{T}$ on $X^{[n+1,n]}\backslash D$. Thus G is trivial outside D. This yields $PD(c_1(G)) = \mu[D]$, where μ is rational. The computation of μ is local, as in Lemma 3. Thus, using the exact sequence (3), we get $\mu = -1$. □

If X is a projective surface, the subring of $H^*(X^{[n]}, \mathbb{Q})$ generated by the classes $\mathrm{ch}_k(E^{[n]})$ (where E runs through all the algebraic vector bundles on X) is called the *ring of algebraic classes of* $X^{[n]}$. If (X, J) is an almost-complex compact four-manifold, we can in the same manner consider the subring of $H^*(X^{[n]}, \mathbb{Q})$ generated by the classes $\mathrm{ch}_k(E^{[n]})$, where E runs through all the complex vector bundles on X. If X is projective, this ring is much bigger than the ring of the algebraic classes. In a forthcoming paper [12], we will show that it is indeed equal to $H^*(X^{[n]}, \mathbb{Q})$ if X is a symplectic compact four-manifold satisfying $b_1(X) = 0$, and we will describe the ring structure of $H^*(X^{[n]}, \mathbb{Q})$.

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Appendix: the decomposition theorem for semi-small maps

In this appendix, we provide Le Potier's unpublished proof of the decomposition theorem for semi-small maps (Theorem 4). For the formalism of the six operations in the derived category of constructible sheaves, we refer the reader to [5] and [14]. Let Z be a complex irreducible quasi-projective variety endowed with a stratification Z_ν. For any positive integer k, we define $U_k = \bigsqcup_{\mathrm{codim}(Z_\nu)\geq k} Z_\nu$. The U_k's form an increasing family of open sets in Z. For any constructible complex $C^\bullet$ on Z, we denote the complex $C^\bullet_{|U_k}$ by $C^\bullet_k$.
Let us recall briefly (mainly to fix the notations) the definition and the basic properties about intersection cohomology needed for the proof of the decomposition theorem. They can be found in [9].

Definition 9. Let $\mathcal{L}$ be a local system of $\mathbb{Q}$–vector spaces on U_0. The intersection complex $IC(\mathcal{L})$ associated to $\mathcal{L}$ with the middle perversity is a bounded constructible complex on Z satisfying the following conditions:

(i) $IC(\mathcal{L})_0 \simeq \mathcal{L}$,
(ii) $\mathcal{H}^i(IC(\mathcal{L})_0) = 0$ if $i > 0$,
(iii) If $j \geq 1$, $k \geq 0$ and $j \geq k$, $\mathcal{H}^j(IC(\mathcal{L})_k) = 0$,
(iv) If $k \geq 1$ and if $i\colon U_k \longrightarrow U_{k+1}$ is the canonical injection, then the adjunction morphism $IC(\mathcal{L})_{k+1} \longrightarrow R\,i_*\,i^{-1}\,IC(\mathcal{L})_{k+1} = R\,i_*IC(\mathcal{L})_k$ is a quasi-isomorphism in degrees at most k.

In the bounded derived category of $\mathbb{Q}$–constructible sheaves on Z, $IC(\mathcal{L})$ is unique up to a unique isomorphism.
For any stratum S of codimension k in Z, let $j_S: S \longrightarrow Z$ be the corresponding inclusion. If $i: U_{k-1} \longrightarrow U_k$ is the canonical injection, we have the adjunction triangle

$$\bigoplus_{S,\,\mathrm{codim} S=k} j_{S*}\, j_S^! IC(\mathcal{L}) \longrightarrow IC(\mathcal{L})_k \longrightarrow R\, i_*\, IC(\mathcal{L})_{k-1} \xrightarrow[+1]{}$$

The conditions (iii) and (iv) imply that $\mathcal{H}^i\left(j_S^! IC(\mathcal{L})\right) = 0$ if $i \leq k$.
The main ingredient in Le Potier's proof is the following lifting lemma:

Lemma 4. *Let D be the derived category of an abelian category $\mathcal{C}$, $A^\bullet$, $B^\bullet$ and $C^\bullet$ be three complexes in $\mathcal{C}$ and $f: B^\bullet \longrightarrow C^\bullet$ be a morphism of complexes such that:*

(i) $A^\bullet$ is concentrated in degrees at most k,
(ii) f induces in cohomology a morphism which is bijective in degrees at most $k-1$, and injective in degree k.

Then:

(i) The morphism $\phi_f: Hom_D(A^\bullet, B^\bullet) \longrightarrow Hom_D(A^\bullet, C^\bullet)$ induced by f is injective,
(ii) The image of ϕ_f consists of morphisms $g: A^\bullet \longrightarrow C^\bullet$ in D such that the induced morphism $\mathcal{H}^k(A^\bullet) \longmapsto \mathcal{H}^k(C^\bullet)$ factors through $\mathcal{H}^k(B^\bullet)$.

Remark 5. When f induces in cohomology a bijective morphism in degrees at most k, Lemma 4 is proved in [9, p. 95].

Proof. Let $M^\bullet$ be the mapping cone of f. Then we have a distinguished triangle $B^\bullet \longrightarrow C^\bullet \longrightarrow M^\bullet \xrightarrow[+1]{}$ and the hypotheses imply that $\mathcal{H}^q(M^\bullet)$ vanishes for $q \leq k-1$. Therefore $\mathrm{Hom}_D(A^\bullet, M^\bullet[-1]) = 0$. Now, the distinguished triangle $\mathrm{RHom}_D(A^\bullet, B^\bullet) \longrightarrow \mathrm{RHom}_D(A^\bullet, C^\bullet) \longrightarrow \mathrm{RHom}_D(A^\bullet, M^\bullet) \xrightarrow[+1]{}$ yields a long exact sequence

$$\mathrm{Hom}_D(A^\bullet, M^\bullet[-1]) \longrightarrow \mathrm{Hom}_D(A^\bullet, B^\bullet) \xrightarrow{\phi_f} \mathrm{Hom}_D(A^\bullet, C^\bullet) \longrightarrow \mathrm{Hom}_D(A^\bullet, M^\bullet)$$

which proves (i).
For (ii), remark that $M^\bullet$ (resp. $A^\bullet$) is concentrated in degrees at least k (resp. at most k), so that $\mathrm{Hom}_D(A^\bullet, M^\bullet) \simeq \mathrm{Hom}_D(\mathcal{H}^k(A^\bullet), \mathcal{H}^k(M^\bullet))$. Thus $\mathrm{Im}\,\phi_f$ consists of the elements g in $\mathrm{Hom}_D(A^\bullet, C^\bullet)$ such that the induced morphism $\mathcal{H}^k(A^\bullet) \longrightarrow \mathcal{H}^k(M^\bullet)$ vanishes. The result follows from the exact sequence $0 \longrightarrow \mathcal{H}^k(B^\bullet) \longrightarrow \mathcal{H}^k(C^\bullet) \longrightarrow \mathcal{H}^k(M^\bullet)$. □

We now turn to the proof of the decomposition theorem.

Proof (of Theorem 4). We construct the quasi-isomorphism between $Rf_*\mathbb{Q}_Y$ and $\bigoplus_{\nu \text{ relevant}} j_{\nu *}\, IC_{\overline{Z}_\nu}(\mathcal{L}_\nu)[-d_\nu]$ by induction on the increasing family of open sets U_i associated to the stratification on Z.
On U_0, the quasi-isomorphism $Rf_{0*}\,\mathbb{Q}_{Y_0} \simeq \mathcal{L}_0$ holds by definition of $\mathcal{L}_0$. If k is a positive integer, assume that we have constructed a quasi-isomorphism $\lambda_{k-1} : \left(Rf_*\mathbb{Q}_Y\right)_{k-1} \xrightarrow{\sim} \bigoplus_{\substack{\nu \text{ relevant} \\ d_\nu \leq k-1}} j_{\nu *}\, IC_{\overline{Z}_\nu}(\mathcal{L}_\nu)[-d_\nu]$. Let us introduce the following notations:

(i) $\mathcal{S} = \bigoplus_{\substack{\nu \text{ relevant} \\ d_\nu \leq k-1}} j_{\nu *}\, IC_{\overline{Z}_\nu}(\mathcal{L}_\nu)[-d_\nu]$,

(ii) $i : U_{k-1} \longrightarrow U_k$ is the injection,

(iii) $A^\bullet = \left(Rf_*\mathbb{Q}_Y\right)_k$, $B^\bullet = \mathcal{S}_k$, $C^\bullet = R\, i_* \mathcal{S}_{k-1}$,

(iv) $f : B^\bullet \longrightarrow C^\bullet$ is the adjunction morphism $\mathcal{S}_k \longrightarrow R\, i_* i^{-1}\mathcal{S}_k = R\, i_*\, \mathcal{S}_{k-1}$.

We have a natural morphism $g : A^\bullet \longrightarrow C^\bullet$ given by the chain of morphisms

$$\left(Rf_*\mathbb{Q}_Y\right)_k \longrightarrow R\, i_* i^{-1}\left(Rf_*\mathbb{Q}_Y\right)_k = R\, i_* \left(Rf_*\mathbb{Q}_Y\right)_{k-1} \xrightarrow{R\, i_*\lambda_{k-1}} R\, i_*\, \mathcal{S}_{k-1}\ .$$

We now check the hypotheses of Lemma 4.

- The complex $\left(Rf_*\mathbb{Q}_Y\right)_k$ is concentrated in degrees at most k. Indeed, the fibers of f over U_k have real dimension at most k, and for every element x of Z, $\left(Rf_*\mathbb{Q}_Y\right)_x = R\Gamma\left(f^{-1}(x), \mathbb{Q}\right)$ by proper base change [5, Th. 2.3.26].
- Let S be a stratum of codimension k in Z and Z_ν be a relevant stratum such that $d_\nu \leq k-1$. Then $S \neq Z_\nu$ and we have either $S \subsetneq Z_\nu$ or $S \cap Z_\nu = \emptyset$ (which is irrelevant to the question). Let $j_S : S \longrightarrow Z$ be the injection of the stratum S in Z and $j_{S,\nu} : S \longrightarrow \overline{Z}_\nu$ be the injection of S in $\overline{Z}_\nu$. Using the cartesian diagram

$$\begin{array}{ccc} S & \xrightarrow{\text{id}} & S \\ {\scriptstyle j_{S,\nu}}\downarrow & & \downarrow{\scriptstyle j_S} \\ \overline{Z}_\nu & \xrightarrow[j_\nu]{} & Z \end{array}$$

we obtain that $j_S^!\, j_{\nu *}\, IC_{\overline{Z}_\nu}(\mathcal{L}_\nu)[-d_\nu] = j_{S,\nu}^!\, IC_{\overline{Z}_\nu}(\mathcal{L}_\nu)[-d_\nu]$. The cohomology sheaf $\mathcal{H}^q\left(j_{S,\nu}^!\, IC_{\overline{Z}_\nu}(\mathcal{L}_\nu)[-d_\nu]\right)$ vanishes if $q - d_\nu \leq \text{codim}_{\overline{Z}_\nu} S = k - d_\nu$, i.e. if $q \leq k$. Thus $\mathcal{H}^q(j_S^!\,\mathcal{S}) = 0$ for $q \leq k$. Let us now write the adjunction triangle

$$j_{S*}\, j_S^!\, \mathcal{S}_k \longrightarrow \mathcal{S}_k \xrightarrow{f} R\, i_{S*}\, i_S^{-1}\mathcal{S}_k \xrightarrow[+1]{}$$

The previous result proves that f is a quasi-isomorphism in degrees at most $k-1$ on S, and then on U_k. In degree k, since $\mathcal{H}^k\left(j_{S*}\, j_S^!\, \mathcal{S}_k\right)=0$, the induced map $\mathcal{H}^k(f)$ is injective. We can also remark that $\mathcal{H}^k(\mathcal{S}_k)$ vanishes. Indeed, on $U_k \cap \overline{Z}_\nu$, all strata have codimension at most $k-d_\nu$, so that $\mathcal{H}^j\left(IC_{\overline{Z}_\nu}(\mathcal{L}_\nu)_{|U_k\cap \overline{Z}_\nu}\right)$ vanishes if $j \geq k-d_\nu$.

– Let us prove that g can be lifted to a morphism $\widetilde{\lambda}_k \colon A^\bullet \longrightarrow B^\bullet$. Since $\mathcal{H}^k(B^\bullet)$ vanishes, the condition (ii) of Lemma 4 means that the map $\mathcal{H}^k(g)$ is identically zero. We will prove a slightly stronger result, namely that the map $\theta \colon \left(R^k f_* \mathbb{Q}_Y\right)_k \longrightarrow R^k i_* \left(Rf_* \mathbb{Q}_Y\right)_{k-1}$ vanishes. Let $F_k = U_k \setminus U_{k-1}$ be the closed set consisting of all k–codimensional strata in Z and let $j \colon F_k \longrightarrow U_k$ be the inclusion. The vanishing of θ is equivalent to the surjectivity of the map $\psi \colon \mathcal{H}^k\left(j_*\, j^!\, Rf_* \mathbb{Q}_Y\right) \longrightarrow \mathcal{H}^k\left(Rf_* \mathbb{Q}_Y\right)_k$. Let Z_ν be a stratum in F_k. We consider the following cartesian diagram:

$$\begin{array}{ccc} Y_\nu & \xrightarrow{\;i_\nu\;} & Y \\ {\scriptstyle f_\nu}\downarrow & & \downarrow{\scriptstyle f} \\ Z_\nu & \xrightarrow[\;j_\nu\;]{} & Z \end{array}$$

If D is the Verdier duality functor, since Y (resp. Z_ν) is rationally smooth (resp. smooth), we get:

$$\begin{aligned} j_\nu^!\, Rf_* \mathbb{Q}_Y &= Rf_{\nu*}\, i_\nu^!\, \mathbb{Q}_Y = Rf_{\nu*}\, i_\nu^!\, \omega_Y[-2\dim Y] \\ &= Rf_{\nu*}\, \omega_{Y_\nu}[-2\dim Y] = D\left(Rf_{\nu*}\mathbb{Q}_{Y_\nu}\right)[-2\dim Y] \\ &= \mathcal{RH}om_{\mathbb{Q}_{Z_\nu}}\left(Rf_{\nu*}\mathbb{Q}_{Y_\nu}, \mathbb{Q}_{Z_\nu}\right)[2\dim Z_\nu - 2\dim Y]. \end{aligned}$$

Now we have $\mathcal{H}^k\left(j_\nu^!\, Rf_* \mathbb{Q}_Y\right) = \mathcal{H}om_{\mathbb{Q}_{Z_\nu}}\left(R^{2\dim Y - 2\dim Z_\nu - k} f_{\nu*}\mathbb{Q}_{Y_\nu}, \mathbb{Q}_{Z_\nu}\right)$ and since $k = \dim Z - \dim Z_\nu$, we obtain

$$\mathcal{H}^k\left(j_\nu^!\, Rf_* \mathbb{Q}_Y\right) = \mathcal{H}om_{\mathbb{Q}_{Z_\nu}}\left(R^k f_{\nu*}\mathbb{Q}_{Y_\nu}, \mathbb{Q}_{Z_\nu}\right) = \begin{cases} \mathcal{L}_\nu^* & \text{if } Z_\nu \text{ is relevant} \\ 0 & \text{otherwise} \end{cases}$$

Remark that the fibers of f_ν are complex projective varieties, so that $\mathcal{L}_\nu \simeq \mathcal{L}_\nu^*$. Thus

$$\mathcal{H}^k\left(j_*\, j^!\, Rf_* \mathbb{Q}_Y\right) \simeq \bigoplus_{\substack{\nu \text{ relevant} \\ \mathrm{codim}(Z_\nu)=k}} j_{\nu*}\mathcal{L}_\nu.$$

This isomorphism can be interpreted in the following way: if we consider the canonical morphism from $\left(Rf_* \mathbb{Q}_Y\right)_k$ to $\left(R^k f_* \mathbb{Q}_Y\right)_k[-k]$, then the associated morphism from $\mathcal{H}^k_{F_k}\Big(Rf_*\,\mathbb{Q}_Y\Big)_k$ to $\mathcal{H}^k_{F_k}\Big(R^k f_*\,\mathbb{Q}_Y[-k]\Big)_k$ is a quasi-isomorphism. Therefore, in the following diagram

$$\begin{array}{ccc}
\mathcal{H}^k_{F_k}\Big(Rf_*\,\mathbb{Q}_Y\Big)_k & \xrightarrow{\ \sim\ } & \mathcal{H}^0_{F_k}\Big(R^k f_*\,\mathbb{Q}_Y\Big)_k \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\sim} \\
\mathcal{H}^k\Big(Rf_*\,\mathbb{Q}_Y\Big)_k & \xrightarrow{\ \sim\ } & \Big(R^k f_*\,\mathbb{Q}_Y\Big)_k
\end{array}$$

the map ψ is an isomorphism; in particular ψ is surjective. This implies that θ vanishes.

The hypotheses of lemma 4 being fulfilled, there exists a canonical morphism $\widetilde{\lambda}_k:\ \Big(Rf_*\,\mathbb{Q}_Y\Big)_k \longrightarrow \mathcal{S}_k$ such that the diagram

$$\begin{array}{ccc}
\Big(Rf_*\,\mathbb{Q}_Y\Big)_k & \longrightarrow & R\,i_*\Big(Rf_*\,\mathbb{Q}_Y\Big)_{k-1} \\
{\scriptstyle\widetilde{\lambda}_k}\downarrow & & \downarrow{\scriptstyle R\,i_*\,\lambda_{k-1}} \\
\mathcal{S}_k & \xrightarrow[f]{} & R\,i_*\,\mathcal{S}_{k-1}
\end{array}$$

commutes. We look now at the following morphism of distinguished triangles

$$\begin{array}{ccccccc}
\bigoplus\limits_{\nu,\,c_\nu=k} j_{\nu *}\,j_\nu^!\Big(Rf_*\,\mathbb{Q}_Y\Big)_k & \longrightarrow & \Big(Rf_*\,\mathbb{Q}_Y\Big)_k & \longrightarrow & R\,i_*\Big(Rf_*\,\mathbb{Q}_Y\Big)_{k-1} & \xrightarrow{+1} & \\
{\scriptstyle j_{\nu *}\,j_\nu^!\,\widetilde{\lambda}_k}\downarrow & & {\scriptstyle\widetilde{\lambda}_k}\downarrow & & {\scriptstyle\simeq}\downarrow{\scriptstyle R\,i_*\,\lambda_{k-1}} & & \\
\bigoplus\limits_{\nu,\,c_\nu=k} j_{\nu *}\,j_\nu^!\,\mathcal{S}_k & \longrightarrow & \mathcal{S}_k & \longrightarrow & R\,i_*\,\mathcal{S}_{k-1} & \xrightarrow[+1]{} &
\end{array}$$

where $c_\nu = \operatorname{codim}_Z(Z_\nu)$.

- Since $j_\nu^!\Big(Rf_*\mathbb{Q}_Y\Big)_k \simeq \mathcal{RH}om_{\mathbb{Q}_{Z_\nu}}\Big(Rf_*\mathbb{Q}_{Y_\nu}, \mathbb{Q}_{Z_\nu}\Big)[-2k],\ j_\nu^!\Big(Rf_*\mathbb{Q}_Y\Big)_k$ is concentrated in degrees at least $2k - d_\nu$. Besides, $d_\nu \leq c_\nu = k$. Thus, the complex $j_\nu^!\Big(Rf_*\mathbb{Q}_Y\Big)_k$ is concentrated in degrees at least k.
- The complex $j_\nu^!\,\mathcal{S}_k$ is concentrated in degrees at least $k+1$. This shows that $\widetilde{\lambda}_k$ is a quasi-isomorphism in degrees at most $k-2$.

If we denote by $A \longrightarrow B \longrightarrow C \xrightarrow{+1}$ and $A' \longrightarrow B' \longrightarrow C' \xrightarrow{+1}$ the two distinguished triangles corresponding to the two lines of the previous diagram, we write down the long cohomology exact sequences and we get another diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow \mathcal{H}^{k-1}(B) & \longrightarrow & \mathcal{H}^{k-1}(C) & \longrightarrow & \mathcal{H}^{k}(A) & \longrightarrow & \mathcal{H}^{k}(B) \\
\downarrow & & \downarrow \simeq & & \downarrow & & \\
0 \longrightarrow \mathcal{H}^{k-1}(B') & \longrightarrow & \mathcal{H}^{k-1}(C') & \longrightarrow & 0 & &
\end{array}$$

We have seen that the map $\mathcal{H}^k(A) \longrightarrow \mathcal{H}^k(B)$ is a quasi-isomorphism. This implies that $\mathcal{H}^{k-1}(B)$ and $\mathcal{H}^{k-1}(C)$ are isomorphic and proves that $\widetilde{\lambda}_k$ is a quasi-isomorphism in degree $k-1$.

If μ_k denotes the natural morphism from $\left(Rf_* \mathbb{Q}_Y\right)_k$ to $\left(R^k f_* \mathbb{Q}_Y\right)_k [-k]$, let us define $\lambda_k = \left(\widetilde{\lambda}_k, \mu_k\right)$. Since $\left(R^k f_* \mathbb{Q}_Y\right)_k [-k] = \bigoplus_{\substack{\nu \text{ relevant} \\ d_\nu = k}} j_{\nu *} \mathcal{L}_\nu[-k]$, λ_k is a morphism from $\left(Rf_* \mathbb{Q}_Y\right)_k$ to $\bigoplus_{\substack{\nu \text{ relevant} \\ d_\nu \leq k}} j_{\nu *} \mathcal{L}_\nu[-k]$ which is a quasi-isomorphism in degrees at most $k-1$ and also in degree k. It is zero in degrees at least k. Therefore λ_k is a quasi isomorphism and the induction step is completed. This finishes the proof of the decomposition theorem. □

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