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Tian's Invariant of the Grassmann Manifold

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ABSTRACT. *We prove that Tian's invariant on the complex Grassmann manifold $G_{p,q}(\mathbb{C})$ is equal to $1/(p+q)$. The method introduced here uses a Lie group of holomorphic isometries which operates transitively on the considered manifolds and a natural imbedding of $(\mathbb{P}^1(\mathbb{C}))^p$ in $G_{p,q}(\mathbb{C})$.*

Résumé. On prouve que l'invariant de Tian sur la grassmannienne $G_{p,q}(\mathbb{C})$ est $1/(p+q)$. La méthode présentée dans cet article utilise un groupe de Lie d'isométries holomorphes qui opère transitivement sur les variétés considérées ainsi qu'un plongement naturel de $(\mathbb{P}^1(\mathbb{C}))^p$ dans $G_{p,q}(\mathbb{C})$.

1. Introduction

On a complex manifold, an hermitian metric h is characterized by the *1-1 symplectic* form ω defined by $\omega = i\, g_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$, where $g_{\lambda\bar{\mu}} = h_{\lambda\bar{\mu}}/2$.

The metric is a *Kähler metric* if ω is closed, i.e., $d\omega = 0$; then M is a *Kähler manifold.*

On a Kähler manifold, we can define the *Ricci form* by $R = i\, R_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$, where $R_{\lambda\bar{\mu}} = -\partial_{\lambda\bar{\mu}} \log |g|$.

A Kähler manifold is *Einstein with factor k* if $R = k\omega$. For instance, choosing a local coordinate system $Z = (z_1, \dots, z_m)$, the projective space $\mathbb{P}_m(\mathbb{C})$ with the Fubini-Study metric $\omega = i\partial\bar{\partial} \log(1 + ||Z||^2)$ is Einstein with factor $m+1$.

On a Kähler manifold M, the *first Chern class* $C^1(M)$ is the cohomology class of the Ricci tensor, that is the set of the forms $R + i\partial\bar{\partial}\varphi$, where φ is C^∞ on M. If there is a form in $C^1(M)$ which is positive (resp. negative, zero), then $C^1(M)$ is *positive* (resp. *negative, zero*). If a Kähler manifold is Einstein, then $C^1(M)$ and k are both positive (resp. negative, zero). In the negative case, it was proved by Aubin [1], see also [4], that there exists a unique Einstein-Kähler metric (E. K. metric) on M. It is so for the zero case too [1, 15]. The question for the positive case is still open: Some manifolds, such as the complex projective space blown up at one point, do not admit an E. K. metric (for obstructions, see [11] and [9]). Aubin [2] and Tian [14] have shown that for suitable values of holomorphic invariants of the metric, there exists an E. K. metric on M.

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For $\omega/2\pi$ in $C^1(M)$, *Tian's invariant* $\alpha(M)$ is the supremum of the set of the real numbers α satisfying the following: There exists a constant C such that the inequality $\int_M e^{-\alpha\varphi} \leq C$ holds for all the C^∞ functions φ with $\omega + i\partial\bar{\partial}\varphi > 0$ and $\sup\varphi \geq 0$, where $\omega = i\, g_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$ is the metric form. Such functions φ are said *ω-admissible.*

In [14], Tian established that if $\alpha(M) > m/(m+1)$, m being the dimension of M, there exists an E. K. metric on M. This condition is not necessary: It does not hold on the projective space, where Tian's invariant is $1/(m+1)$.

In the same article, Tian introduces a more restrictive invariant $\alpha_G(M)$, considering only the admissible functions φ invariant by the action of a compact group G of holomorphic isometries. The sufficient condition for the existence of an E. K. metric on M remains $\alpha_G(M) > m/(m+1)$; it is more easily satisfied if the group G is rich enough.

In many cases, the group G is a nondiscrete Lie group. The invariant $\alpha_G(M)$ can be computed using subharmonic functions methods and the maximum principle (for effective examples, see [5, 6, 7, 8, 13]).

In this article, we prove the following theorem.

Theorem 1.1. *Tian's invariant on $G_{p,q}(\mathbb{C})$ is given by $\alpha\left(G_{p,q}(\mathbb{C})\right) = 1/(p+q)$.*

This generalizes the known result on $\mathbb{P}^m(\mathbb{C})$ [14], see also [3]. Let us also mention that Tian's invariant has been computed on $\mathbb{P}^m(\mathbb{C})$ blown up at one point and on certain Fermat hypersurfaces using Hörmander L^2 estimates for the $\bar{\partial}$-equation [14].

We first compute the volume element of the metric $\mathcal{G}_{p,q}$; then we will establish some general preliminary results concerning Tian's invariant as well as imbeddings of $\{\mathbb{P}^1(\mathbb{C})\}^p$ in $G_{p,q}(\mathbb{C})$ which allow us to deduce $\alpha\left(G_{p,q}(\mathbb{C})\right)$ from $\alpha\left(\mathbb{P}^1(\mathbb{C})\right)$.

2. Basic properties of the Grassmann manifold

We propose here a short survey of the properties of the Grassmann manifold (for more details, see [10]). We denote by $G_{p,q}(\mathbb{C})$ the set of the subspaces of dimension p in $\mathbb{C}^{p+q}$; in particular, $G_{1,m}(\mathbb{C})$ is the complex projective space of dimension m. It is known (see [3]) that on $\mathbb{P}_m(\mathbb{C})$, the Fubini-Study metric is Einstein with factor $m+1$ and that Tian's invariant is $1/(m+1)$. Now, let $M^*(p+q, p)$ be the set of the matrices of rank p in $M_{p+q,p}(\mathbb{C})$. The group $Gl_p(\mathbb{C})$ acts by multiplication on the right on $M^*(p+q, p)$. More precisely, $\left(M^*(p+q, p),\ \pi,\ G_{p,q}(\mathbb{C})\right)$ is a principal fiber bundle with group $Gl_p(\mathbb{C})$. The group $Gl_{p+q}(\mathbb{C})$ acts by multiplication on the left on $M^*(p+q, p)$ and induces an action on $G_{p,q}(\mathbb{C})$; so does the unitary group $U(p+q)$. These groups act transitively on $G_{p,q}(\mathbb{C})$, which shows that $G_{p,q}(\mathbb{C})$ is compact.

We denote by $\mathcal{I}$ the set of all increasing-ordered subsets of p elements in $\{1, \ldots, p+q\}$. Let P be an element of $M^*(p+q, p)$, $P = (p_{ij})_{\substack{1\leq i\leq p+q \\ 1\leq j\leq p}}$. By Cauchy-Binet formula we get: $\det\left({}^tP\,\overline{P}\right) = \sum_{I\in\mathcal{I}} |\det m_I(P)|^2$, where $m_I(P)$ is the matrix $(p_{ij})_{\substack{i\in I \\ 1\leq j\leq p}}$. The form ω, where $\omega = i\,\partial\bar{\partial}\,\log\det\left({}^tP\,\overline{P}\right)$, is invariant by the action of $Gl_p(\mathbb{C})$ on $M^*(p+q, p)$, and so it projects onto a form $\mathcal{G}_{p,q}$. The metric $\mathcal{G}_{p,q}$ is a Kähler metric form on $G_{p,q}(\mathbb{C})$. For $p = 1$, this metric on $G_{1,m}(\mathbb{C})$ is the Fubini-Study metric on the complex projective space. The action of the unitary group $U(p+q)$ on $G_{p,q}(\mathbb{C})$ preserves the metric $\mathcal{G}_{p,q}$ so that $U(p+q)$ is a group of holomorphic isometries which operates transitively on $G_{p,q}(\mathbb{C})$.

For I in $\mathcal{I}$, let U_I be the set of the matrices P in $M^*(p+q,\, p)$ such that $\det(m_I(P))$ is nonzero. Then $\pi(U_I)$ is a coordinate open set on $G_{p,q}(\mathbb{C})$, the matrix Z_I in $M_{q,p}(\mathbb{C})$ is the coordinate, the inverse of the chart φ_I sends $M^*(p+q,\, p)$ onto $\pi(U_I)$ and we have $m_I\big(\varphi_I^{-1}(Z_I)\big) = I^{(p)}$ where $I^{(p)}$ is the $p \times p$ identity matrix, and $m_{I^c}\big(\varphi_I^{-1}(Z_I)\big) = Z_I$.

Lemma 2.1. *For I in $\mathcal{I}$, let λ_I be the map from $\pi(U_I)$ to $\mathbb{R}_+$ defined by*

$$\lambda_I(Z_I) = \big|\det\big(\mathrm{Id} + {}^t Z_I \overline{Z}_I\big)\big|^{-(p+q)} .$$

Then $(\lambda_I)_{I\in\mathcal{I}}$ are the components of a maximal differential form η on $G_{p,q}(\mathbb{C})$, namely:

$$\eta = \lambda_I (i/2)^{pq} \left(dZ \wedge d\overline{Z}\right)_I .$$

Proof. It suffices to show that the following transformation rule holds:

$$\text{for every } I,\, \tilde{I} \text{ in } \mathcal{I},\ \lambda_I \text{ is equal to } \lambda_{\tilde{I}} \times \left|\det \frac{\partial Z_{\tilde{I}}}{\partial Z_I}\right|^2 \quad \text{on } \pi(U_I) \cap \pi(U_{\tilde{I}}) .$$

Let P_I be the matrix $\varphi_I^{-1}(Z_I)$. Then $P_I\{m_{\tilde{I}}(P_I)\}^{-1} = P_{\tilde{I}}$, so $Z_{\tilde{I}} = m_{\tilde{I}^c}(P_I)\,\{m_{\tilde{I}}(P_I)\}^{-1}$. The differential of the map which sends Z_I on P_I is the map which sends H on $\breve{H}$, where $m_{I^c}(\breve{H}) = H$ and $m_I(\breve{H}) = 0$. The change of charts sending Z_I on $Z_{\tilde{I}}$, we obtain

$$\begin{aligned} D\,Z_{\tilde{I}}(H) &= m_{\tilde{I}^c}(\breve{H})\,\{m_{\tilde{I}}(P_I)\}^{-1} - m_{\tilde{I}^c}(P_I)\,\{m_{\tilde{I}}(P_I)\}^{-1}\, m_{\tilde{I}}(\breve{H})\,\{m_{\tilde{I}}(P_I)\}^{-1} \\ &= \big(m_{\tilde{I}^c}(\breve{H}) - \gamma\, m_{\tilde{I}}(\breve{H})\big)\,\alpha^{-1} , \\ \text{where} \quad \alpha &= m_{\tilde{I}}(P_I),\ \beta = m_{\tilde{I}^c}(P_I) \text{ and } \gamma = \beta\,\alpha^{-1} . \end{aligned}$$

Let us define a map u from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ by $u(H) = m_{\tilde{I}^c}(\breve{H}) - \gamma\, m_{\tilde{I}}(\breve{H})$. We can choose $I = \{q+1, \ldots, q+p\}$ and $\tilde{I} = \{1, \ldots, r\} \cup \{q+1+r, \ldots, q+p\}$, where $0 \le r \le \inf(p, q)$. We define the $k \times l$ matrix $E_{i,j}^{(k\times l)}$ by $\big(E_{i,j}^{(k\times l)}\big)_{\lambda\mu} = \delta_{i\lambda}\,\delta_{j\mu}$. We have

$$\begin{aligned} & m_{\tilde{I}}\big(\breve{E}_{i,j}^{(q\times p)}\big) = E_{i,j}^{(p\times p)} \quad \text{if} \quad i \le r, \text{ and } 0 \text{ if } \quad i > r , \\ \text{and} \quad & m_{\tilde{I}^c}\big(\breve{E}_{i,j}^{(q\times p)}\big) = E_{i-r,j}^{(q\times p)} \quad \text{if} \quad i > r, \quad \text{and} \quad 0 \text{ if } \quad i \le r. \text{ Hence,} \\ & \big(\gamma\, m_{\tilde{I}}\big(\breve{E}_{i,j}^{(q\times p)}\big)\big)_{\alpha\beta} = \gamma_{\alpha i}\, m_{\tilde{I}}\big(\breve{E}_{i,j}^{(q\times p)}\big)_{ij}\, \delta_{j\beta} = \gamma_{\alpha i}\, \delta_{j\beta} \quad \text{if} \quad i \le r, \text{ and } 0 \text{ elsewhere} . \end{aligned}$$

Now the map which sends H to $\gamma\, m_{\tilde{I}}(\breve{H})$ can be restricted if $1 \le j \le p$ to the span B_j of the $(E_{i,j})_{1\le i\le q}$. The r first columns of its matrix are those of γ, the others are 0. The map which sends H to $\gamma\, m_{\tilde{I}^c}(\breve{H})$ maps also B_j into itself. The right upper block of its matrix is $I^{(q-r)}$, the other elements are 0. This allows us to compute the matrix of the restriction of u to B_j, whose determinant is $(-1)^{r\times(q-r)} \det(\gamma_{ij})_{\substack{q-r+1\le i\le q \\ 1\le j\le r}}$. So $\det u = (-1)^{p\times r\times(q-r)} \Big[\det(\gamma_{i\,j})_{\substack{q-r+1\le i\le q \\ 1\le j\le r}}\Big]^p$. For $1 \le i \le q$, let C_i be the span of the $(E_{i,j})_{1\le j\le p}$. Each C_i is stable by the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to $H\,\alpha^{-1}$. The matrix of the restriction is α^{-1}, so the determinant of the map is $(\det\alpha)^{-q}$. Hence,

$$\big|\det\, DZ_{\tilde{I}}(H)\big|^2 = \Big|\det\,(\gamma_{i,j})_{\substack{q-r+1\le i\le q \\ 1\le j\le r}}\Big|^{2p} \times \big|\det\alpha\big|^{-2q} .$$

Let A be the right $r \times r$ upper block of α. The left $(p-r) \times (p-r)$ lower block of α is $I^{(p-r)}$ and the right $(p-r) \times r$ lower block is 0, so $\det\alpha = (-1)^{r(p-r)} \det A$. The left $r \times (p-r)$ lower block of β is 0, the right $r \times r$ block is $I^{(r)}$ so that the left $r \times r$ lower block of γ is A^{-1}.

From this we deduce $\left|\det DZ_{\tilde{I}}(H)\right|^2 = \left|\det\alpha\right|^{-2(p+q)}$. Since $P_I\,\alpha^{-1} = P_{\tilde{I}}$, we have

$$\lambda_{\tilde{I}} = \left|\det\left({}^t P_{\tilde{I}}\overline{P}_{\tilde{I}}\right)\right|^{-(p+q)} = \left|\det\alpha\right|^{2(p+q)}\lambda_I = \left|\det\frac{\partial Z_{\tilde{I}}}{\partial Z_I}\right|^{-2}\lambda_I\,. \qquad \square$$

Lemma 2.2. *The unitary group $U(p+q)$ preserves η.*

Proof. We call I the set $\{q+1,\dots,q+p\}$. We define P_I in $\pi(U_I)$ by $P_I = \varphi_I^{-1}(Z_I)$. Let U be an element in $U(p+q)$ such that $m_I(UP_I)$ is invertible. Let $\tilde{P}_I = UP_I\left\{m_I(UP_I)\right\}^{-1}$ and $\tilde{Z}_I = m_{I^c}(\tilde{P}_I)$. We have $\tilde{Z}_I = m_{I^c}(U)\,P_I\left\{m_I(U)P_I\right\}^{-1}$. So

$$D\tilde{Z}_I(H) = m_{I^c}(U)\left[\check{H}\left\{m_I(U)P_I\right\}^{-1} - P_I\left\{m_I(U)P_I\right\}^{-1}m_I(U)\,\check{H}\left\{m_I(U)P_I\right\}^{-1}\right].$$

Thus, $D\tilde{Z}_I(H) = X\check{H}\delta^{-1}$, where $\delta = m_I(U)P_I$ and $X = m_{I^c}(U)\left[I^{(p+q)} - P_I\,\delta^{-1}\,m_I(U)\right]$. Let X_1 be the $q\times q$ matrix of the q first columns of X. Then, $X\check{H} = X_1H$ and we get $D\tilde{Z}_I(H) = X_1H\delta^{-1}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to $H\delta^{-1}$ is $(\det\delta)^{-q}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to X_1H is $(\det X_1)^p$, so $\det D\tilde{Z}_I = (\det X_1)^p\,(\det\delta)^{-q}$. We divide U into four blocks:

$$U = \begin{pmatrix} U_q & U_{q,p} \\ U_{p,q} & U_p \end{pmatrix},\quad U_q\in M_q(\mathbb{C}),\ U_p\in M_p(\mathbb{C}),\ U_{p,q}\in M_{p,q}(\mathbb{C}),\ U_{q,p}\in M_{q,p}(\mathbb{C})\,.$$

Then $\delta = U_{p,q}\,Z_I + U_p$, so $X_1 = U_q - \left(U_q\,Z_I + U_{q,p}\right)\left(U_{p,q}\,Z_I + U_p\right)^{-1}U_{q,p}$. Let Z in $M_{p+q,p+q}(\mathbb{C})$ be the matrix with blocks $Z_q = I^{(q)}$, $Z_{p,q} = 0$, $Z_{q,p} = Z_I$, $Z_p = I^{(p)}$, the notations being the same as above. Writing $\det U = \det(UZ)$ and using the column transformation $C_1 \leftarrow C_1 - C_2\left(U_{p,q}\,Z_I + U_p\right)^{-1}U_{p,q}$ where C_1 is made of the first q columns and C_2 of the remaining ones, we get

$$\det U = \det\left[U_q - \left(U_q\,Z_I + U_{q,p}\right)\left(U_{p,q}\,Z_I + U_p\right)^{-1}U_{p,q}\right]\times\det\left(U_{p,q}Z_I + U_p\right).$$

Hence, $\left|\det D\tilde{Z}_I\right|^2 = \left|\det\delta\right|^{-2(p+q)}$. We have $\tilde{P}_I = AP_I\delta^{-1}$, so

$$\lambda_{\tilde{I}} = \det\left({}^t\tilde{P}_I\,\overline{\tilde{P}}_I\right)^{-(p+q)} = \det\left({}^tP_I\,\overline{P}_I\right)^{-(p+q)}\times\left|\det\delta\right|^{2(p+q)} = \lambda_I\left|\det D\tilde{Z}_I\right|^{-2},$$

which proves the result. $\square$

Proposition 2.3.

(1) $dV\left(\mathcal{G}_{p,q}\right) = \eta$.

(2) *If* $I\in\mathcal{I}$, $\left|\mathcal{G}_{p,q}\right|_I = \left\{\det\left(I^{(p)} + {}^tZ_I\overline{Z}_I\right)\right\}^{-(p+q)}$.

(3) $\mathcal{R}\left(\mathcal{G}_{p,q}\right) = (p+q)\mathcal{G}_{p,q}$.

Proof.

(1) Let I in $\mathcal{I}$. It is easy to compute $\mathcal{G}_{p,q}$ at the point $Z_I = 0$: $\mathcal{G}_{p,q}(H,K) = Tr(H\overline{\overline{K}})$. Then $dV(\mathcal{G}_{p,q})_{|Z_I=0} = \left(i/2\right)^{pq}\left(dZ\wedge d\overline{Z}\right)_I = \eta_{|Z_I=0}$. Since $dV(\mathcal{G}_{p,q})$ and η are invariant by the transitive action of $U(p+q)$, we have $dV(\mathcal{G}_{p,q}) = \eta$.

(2) Since $dV(\mathcal{G}_{p,q}) = \left|\mathcal{G}_{p,q}\right|_I(i/2)^{pq}\left(dZ\wedge d\overline{Z}\right)_I$, property (1) gives the result.

(3) Remark that $\mathcal{G}_{p,q} = i\,\partial\overline{\partial}\,\log\left\{\det(I^{(p)} + {}^tZ_I\overline{Z}_I)\right\}$. Since $\mathcal{R}\left(\mathcal{G}_{p,q}\right) = -i\,\partial\overline{\partial}\,\log\left|\mathcal{G}_{p,q}\right|_I$, we obtain $\mathcal{R}\left(\mathcal{G}_{p,q}\right) = (p+q)\mathcal{G}_{p,q}$, which expresses that $\mathcal{G}_{p,q}$ is Einstein, with factor $p+q$. $\square$

3. Some general results about Tian's invariant

3.1. Tian's invariant with a normalization on a finite set

If X is a manifold, we will denote by μ_X a measure on X compatible with the manifold structure.

Theorem 3.1. *Let M be a compact Kähler manifold and G a compact Lie group of holomorphic isometries. Let $\Delta_n = \{P_1, \dots, P_n\}$ be a finite subset of M. Let $\alpha(\omega)$ (resp. $\alpha_{\Delta_n}(\omega)$) be the supremum of the set of the nonnegative real numbers α satisfying the condition: There exists a constant C such that the inequality $\int_M e^{-\alpha\varphi} \le C$ holds for all the ω-admissible functions φ with $\sup\varphi \ge 0$ (resp. with $\varphi(P_i) \ge 0$ for $1 \le i \le n$). Suppose in addition that the orbit of each P_i under the action of G has positive measure. Then $\alpha(\omega) = \alpha_{\Delta_n}(\omega)$.*

We first establish a few lemmas which will be useful for the proof.

Lemma 3.2. *Let $(\varphi_n)_{n\ge0}$ be a sequence of admissible functions with nonnegative maxima. Then there exists a subset Ω of M, with $\mu_M(\Omega) = \mu_M(M)$, and a subsequence φ_{n_k} of φ_n, such that for every p in Ω, the sequence $(\varphi_{n_k}(p))_{k\ge0}$ has a finite lower bound (depending on p).*

Proof. It is sufficient to assume that φ_n has null maxima. Let Q_n be a point such that $\varphi_n(Q_n)$ vanishes. Green's formula runs as follows:

$$\varphi_n(Q_n) = \frac{1}{V}\int_M \varphi_n + \int_M G(Q_n, R)\,\Delta\varphi_n(R)\,dV(R)\,,$$

with $G(Q, R) \ge 0$ and $\int_M G(Q, R)\,dV(R) = C$, where C is a positive constant (see [4]). Since φ_n is admissible, $\Delta\varphi_n$ is less than m, m being the dimension of M. Thus, $\int_M |\varphi_n| \le C\,m\,V$. Furthermore, $\int_M \Delta\varphi_n = 0$, so $\int_M |\Delta\varphi_n| = 2\int_{\{\Delta\varphi_n>0\}} \Delta\varphi_n \le 2mV$. For every Q in M, we have $\nabla\varphi_n(Q) = \int_M \nabla_Q G(Q, R)\Delta\varphi_n(R)\,dv(R)$, so that

$$\int_M |\nabla\varphi_n| \le \int_M \left[\int_M |\nabla_Q G(Q, R)|\,dv(Q)\right]|\Delta\varphi_n(R)|\,dv(R) \le 2m\widetilde{C}V\,,$$

since $\int_M |\nabla_Q G(Q, R)|\,dv(Q)$ is a continuous, hence a bounded function on M. Thus, $(\varphi_n)_{n\ge0}$ is bounded in the Sobolev space $H^{1,1}(M)$. By Kondrakov's theorem, we can extract from $(\varphi_n)_{n\ge0}$ a subsequence which converges in $L^1(M)$, and after another extraction we can suppose that this sequence converges almost everywhere to a function φ of $L^1(M)$. Since φ is finite almost everywhere, we get the result. □

Lemma 3.3. *Let $(\varphi_n)_{n\ge0}$ be a sequence of admissible functions with nonnegative maxima and suppose that there exists a compact group G of holomorphic isometries of M such that the orbit of each P_i has positive measure. Let $\Phi : G \to \mathbb{R} \cup \{-\infty\}$ be the map defined by $\Phi(g) = \inf_{\Delta_n} \inf_{k\ge0} (\varphi_k \circ g)$. Then there exists g in G such that $\Phi(g)$ is finite.*

Proof. Suppose that $\Phi \equiv -\infty$. For $i = 1, \dots, n$, let A_i be the set of the g in G such that

$\inf_{k\geq 0}(\varphi_k\circ g)(P_i)=-\infty$. The sets A_i are measurable and $\cup_{i=1}^n A_i=G$, so there exists i such that A_i has positive measure. From Lemma 3.2, $A_i.P_i$ is a subset of Ω^c. Since Ω and M have the same measure, the measure of $A_i.P_i$ vanishes. Let u_i be the map from G to M which sends g to $g(P_i)$. Then u_i has constant rank on G. Indeed, $u_i\circ L(g)=\sigma_g\circ u_i$, where $L(g)$ is the left translation by g and σ_g the map from M to M which sends x to $g.x$. Since $G.P_i$ has positive measure, u_i is a submersion on G, so that $u_i(A_i)$ has positive measure. This is a contradiction since $u_i(A_i)=A_i.P_i$. □

We can now prove Theorem 3.1.

Proof. It is clear that $\alpha(\omega)\leq\alpha_{\Delta_n}(\omega)$. Conversely, let $\varepsilon>0$. There exists a sequence $(\varphi_n)_{n\geq 0}$ of admissible functions with positive maxima such that $\int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_k}$ goes to infinity as k goes to infinity. Replacing φ_n by $\varphi_n-\sup\varphi_n$, we can take $\sup\varphi_n=0$. First we apply Lemma 3.2. For the sake of simplicity, we take $\varphi_{n_k}=\varphi_k$. From Lemma 3.3, there exists an element g in G such that $\Phi(g)$ is finite; we define Ψ_k by $\Psi_k=\varphi_k\circ g-\Phi(g)$. Since g is an isometry, Ψ_k is ω-admissible, and from the very definition of Φ, $\Psi_k(P_i)$ is nonnegative. Furthermore, $\int_M e^{-(\alpha(\omega)+\varepsilon)\Psi_k}=e^{(\alpha(\omega)+\varepsilon)\Phi(g)}\int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_k}$. This proves that $\int_M e^{-(\alpha(\omega)+\varepsilon)\Psi_k}$ goes to infinity as k goes to infinity. Then, $\alpha_{\Delta_n}(\omega)\leq\alpha(\omega)+\varepsilon$. This inequality holds for every positive ε, and so $\alpha_{\Delta_n}(\omega)\leq\alpha(\omega)$. □

3.2. Tian's invariant on a product

For a Kähler form ω on a compact Kähler manifold M, $\alpha(\omega)$ is defined as in Theorem 3.1.

Proposition 3.4. *Let $(M_i)_{1\leq i\leq n}$ be compact Kähler manifolds with metric forms $(\omega_i)_{1\leq i\leq n}$. We endow the product $M_1\times\cdots\times M_n$ with the metric $\omega_1\oplus\cdots\oplus\omega_n$. Then $\alpha(\omega_1\oplus\cdots\oplus\omega_n)=\inf_{1\leq i\leq n}\alpha(\omega_i)$.*

Proof. It suffices to make the proof when $n=2$, the general result will follow by induction.
(1) Suppose that $\alpha(\omega_1)\leq\alpha(\omega_2)$, and let $\varepsilon>0$. There exists a sequence $(\varphi_n)_{n\geq 0}$ of ω_1-admissible functions on M_1 with positive maxima such that $\int_{M_1} e^{-(\alpha(\omega_1)+\varepsilon)\varphi_n}$ goes to infinity when n goes to infinity. We define ψ_n on $M_1\times M_2$ by $\psi_n(m_1,m_2)=\varphi_n(m_1)$. Thus, ψ_n is $(\omega_1\oplus\omega_2)$-admissible on $M_1\times M_2$, with positive maximum, and $\int_{M_1\times M_2} e^{-(\alpha(\omega_1)+\varepsilon)\psi_n}=V(M_2)\int_{M_1} e^{-(\alpha(\omega_1)+\varepsilon)\varphi_n}$, so that $\int_{M_1\times M_2} e^{-(\alpha(\omega_1)+\varepsilon)\psi_n}$ goes to infinity when n goes to infinity. We have therefore $\alpha(\omega_1\oplus\omega_2)\leq\alpha(\omega_1)+\varepsilon$. This yields $\alpha(\omega_1\oplus\omega_2)\leq\alpha(\omega_1)$.
(2) Let us now prove the opposite inequality. Let α be a real number such that $\alpha<\inf(\alpha(\omega_1),\alpha(\omega_2))$ and φ an $(\omega_1\oplus\omega_2)$-admissible function on $M_1\times M_2$. If m_2 is in M_2, the function which sends m_1 to $\varphi(m_1,m_2)$ is ω_1-admissible. The same holds for M_1. Let (u,v) in $M_1\times M_2$ be such that $\varphi(u,v)\geq 0$. Then

$$\int_{M_1\times M_2} e^{-\alpha\varphi(m_1,m_2)}\,dV_1\,dV_2=\int_{M_1} e^{-\alpha\varphi(m_1,v)}\left(\int_{M_2} e^{-\alpha[\varphi(m_1,m_2)-\varphi(m_1,v)]}\,dV_2\right)dV_1$$
$$\leq C_2\int_{M_1} e^{-\alpha\varphi(m_1,v)}\,dV_1\leq C_1\,C_2\,.$$

Thus, $\alpha \le \alpha(\omega_1 \oplus \omega_2)$ and we get $\inf(\alpha(\omega_1), \alpha(\omega_2)) \le \alpha(\omega_1 \oplus \omega_2)$. □

3.3. Tian's invariant on $G_{p,q}(\mathbb{C})$

Since there is a natural duality isomorphism between $G_{p,q}(\mathbb{C})$ and $G_{q,p}(\mathbb{C})$, we can assume that $p \le q$ without loss of generality.

Imbedding of $\{\mathbb{P}^1(\mathbb{C})\}^p$ into $G_{p,q}(\mathbb{C})$ when $p \le q$ For w in $\mathbb{C}^{p(q-1)}$, $w = (w_{i,j})_{\substack{1\le i\le q\\ 1\le j\le p\\ i\ne j}}$, we define the map $\tilde{\rho}_w$ from $\{\mathbb{C}^2 \setminus (0,0)\}^p$ to $M_{p+q,p}(\mathbb{C})$ by

$$\tilde{\rho}_w\Big((\lambda_i, \mu_i)_{1\le i\le p}\Big) = \begin{cases} \lambda_i\,\delta_{ij} & \text{if} \quad i \le p \\ w_{i-p,j}\,\lambda_j & \text{if} \quad i > p \text{ and } i \ne j+p \\ \mu_i & \text{if} \quad i > p \text{ and } i = j+p \end{cases}.$$

We make, for $p+1 \le i \le p+q$, the following row transformations: $L_i \leftarrow L_i - \sum_{\substack{1\le j\le p\\ i\ne j+p}} w_{i-p,j}\,L_j$.

We get a matrix $(c_{ij})_{\substack{1\le i\le p+q\\ 1\le j\le p}}$ with $c_{ij} = \delta_{ij}\,\lambda_i$ if $1 \le i \le p$ and $c_{ij} = \delta_{i-p,j}\,\mu_j$ if $p+1 \le i \le p+q$, which has rank p. $\tilde{\rho}_w$ induces a map from $\{\mathbb{P}^1(\mathbb{C})\}^p$ into $G_{p,q}(\mathbb{C})$ as shown on the following diagram, where γ is the projection of the principal fiber bundle $\{\mathbb{C}^2 \setminus (0,0)\}^p$ onto $\{\mathbb{P}^1(\mathbb{C})\}^p$. Remark that $\tilde{\rho}_w$ sends $[0,1] \times \cdots \times [0,1]$ onto $\pi(A)$, where $m_{\{p+1,\ldots,2p\}}(A) = I^{(p)}$ and $m_{\{p+1,\ldots,2p\}^c}(A) = 0^{(q\times p)}$.

$$\begin{array}{ccc} \{\mathbb{C}^2 \setminus (0,0)\}^p & \xrightarrow{\tilde{\rho}_w} & M^*(p+q,p) \\ \gamma\big\downarrow & & \big\downarrow\pi \\ \{\mathbb{P}^1(\mathbb{C})\}^p & \xrightarrow[\rho_w]{} & G_{p,q}(\mathbb{C}) \end{array}$$

We have

$$\begin{aligned} (\pi \circ \tilde{\rho}_w)^*(\mathcal{G}_{p,q}) &= i\,\partial\bar{\partial}\,\log\Big(\det({}^t\tilde{\rho}_w\bar{\tilde{\rho}}_w)\Big) \\ &= i\,\partial\bar{\partial}\,\log\left(\frac{\det\left({}^t\tilde{\rho}_w\bar{\tilde{\rho}}_w\right)}{\prod_{k=1}^{p}\left(|\lambda_k|^2 + |\mu_k|^2\right)}\right) + \sum_{k=1}^{p} i\,\partial\bar{\partial}\,\log\left(|\lambda_k|^2 + |\mu_k|^2\right) \\ &= i\,\partial\bar{\partial}\,\log\tilde{\Phi} + \gamma^*\left(FS_1 \oplus \cdots \oplus FS_1\right), \end{aligned}$$

where FS_1 is the Fubini-Study metric on $\mathbb{P}^1(\mathbb{C})$. $\tilde{\Phi}$ is invariant by the action of the structural group $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$, so it induces a map Φ from $\{\mathbb{P}^1(\mathbb{C})\}^p$ into $\mathbb{C}$. Note that $\Phi([0,1] \times \cdots \times [0,1]) = 1$. Then $(\pi \circ \tilde{\rho}_w)^*(\mathcal{G}_{p,q}) = \pi^*(i\,\partial\bar{\partial}\,\log\Phi + FS_1 \oplus \cdots \oplus FS_1)$, so that $\rho_w^*(\mathcal{G}_{p,q}) = i\,\partial\bar{\partial}\,\log\Phi + FS_1 \oplus \cdots \oplus FS_1$.

Lower bound of $\alpha(\mathcal{G}_{p,q})$ For I in $\mathcal{I}$, we define P_I by $m_I(P_I) = I^{(p)}$ and $m_{I^c}(P_I) = 0^{(q\times p)}$. If $n = \binom{p+q}{p}$, we set $\Delta_n = \{P_I\}_{I\in\mathcal{I}}$. Since $U(p+q)$ is a transitive group of holomorphic isometries of $G_{p,q}(\mathbb{C})$, we know from Proposition 3.1 that $\alpha(\mathcal{G}_{p,q}) = \alpha_{\Delta_n}(\mathcal{G}_{p,q})$. We set $I = \{p+1, \dots, 2p\}$. Let φ be an admissible function on $G_{p,q}(\mathbb{C})$, nonnegative on Δ_n. The last equality of the precedent section shows that the function $\varphi\circ\rho_w + \log\Phi$ is $(FS_1\oplus\cdots\oplus FS_1)$-admissible for every w in $\mathbb{C}^{p(q-1)}$. Furthermore, $(\varphi\circ\rho_w + \log\Phi)$ sends $[0,1]\times\cdots\times[0,1]$ to the nonnegative number $\varphi(P_I)$. It is known that $\alpha(FS_1) = 1$ (see [3]). Proposition 3.4 yields $\alpha(FS_1\oplus\cdots\oplus FS_1) = 1$.

Let α be a real number such that $\alpha < 1$. There exists a constant C, independent of φ, such that $\int_{\{\mathbb{P}^1(\mathbb{C})\}^p} e^{-\alpha\varphi\circ\rho_w}\,\Phi^{-\alpha} \le C$. We define the map F_I from $\pi(U_I)$ to $\mathbb{R}_+$ by $F_I(Z_I) = \det\left(\mathrm{Id} + {}^t Z_I \overline{Z}_I\right)$. On $\{\mathbb{P}^1(\mathbb{C})\}^p$, we work with the coordinates $\mu_1, \dots, \mu_p$ in the chart $\lambda_1 = \cdots = \lambda_p = 1$. Thus,

$$\Phi(\mu) = \frac{F_I\circ\rho_w(\mu)}{\prod_{k=1}^{p}(1+|\mu_k|^2)}, \quad \text{so that} \quad \int_{\mu\in\mathbb{C}^p} e^{-\alpha\varphi\circ\rho_w(\mu)} \frac{dV_\mu(\mathbb{C}^p)}{\prod_{k=1}^{p}(1+|\mu_k|^2)^{2-\alpha}\left(F_I\circ\rho_w(\mu)\right)^\alpha} \le C\,.$$

We have the inequality $1 + \sum_{i=1}^{q}\sum_{j=1}^{p}|Z_{ij}|^2 \le F_I(Z_I)$. In particular, for every k in $\{1,\dots,p\}$, $1+|\mu_k|^2 \le F_I\circ\rho_w(\mu)$, and $F_I\circ\rho_w(\mu) \ge 1 + \sum_{\substack{1\le i\le q\\ 1\le j\le p\\ i\ne j}} |w_{ij}|^2$. Thus, for $\kappa > 0$ and $w\in\mathbb{C}^{p(q-1)}$,

$$\frac{\prod_{k=1}^{p}(1+|\mu_k|^2)^{2-\alpha}}{\left(F_I\circ\rho_w(\mu)\right)^{\kappa+p+q-\alpha}} \le \frac{1}{\left(F_I\circ\rho_w(\mu)\right)^{\kappa-p+q+\alpha(p-1)}} \le \frac{1}{\left(1+\sum_{\substack{1\le i\le q\\ 1\le j\le p\\ i\ne j}}|w_{ij}|^2\right)^\kappa} = \frac{1}{\left(1+\|w\|^2\right)^\kappa}.$$

We have, according to Proposition 2.3,

$$\int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^\kappa} = \int_{w\in\mathbb{C}^{p(q-1)}}\int_{\mu\in\mathbb{C}^p} \frac{e^{-\alpha\varphi\circ\rho_w(\mu)}}{\left(F_I\circ\rho_w(\mu)\right)^{\kappa+p+q}}\, dV_\mu(\mathbb{C}^p)\, dV_w(\mathbb{C}^{p(q-1)})$$

$$= \int_{w\in\mathbb{C}^{p(q-1)}}\int_{\mu\in\mathbb{C}^p} \left(\frac{e^{-\alpha\varphi\circ\rho_w(\mu)}}{\prod_{k=1}^{p}(1+|\mu_k|^2)^{2-\alpha}\left(F_I\circ\rho_w(\mu)\right)^\alpha}\right)$$

$$\times \frac{\prod_{k=1}^{p}(1+|\mu_k|^2)^{2-\alpha}}{\left(F_I\circ\rho_w(\mu)\right)^{\kappa+p+q-\alpha}}\, dV_\mu(\mathbb{C}^p)\, dV_w(\mathbb{C}^{p(q-1)})$$

$$= \int_{w \in \mathbb{C}^{p(q-1)}} \left(\int_{\mu \in \mathbb{C}^p} \frac{e^{-\alpha\varphi \circ \rho_w(\mu)}}{\prod_{k=1}^{p} \left(1 + |\mu_k|^2\right)^{2-\alpha} \left(F_I \circ \rho_w(\mu)\right)^\alpha} dV_\mu\left(\mathbb{C}^p\right) \right) \times \frac{dV_w\left(\mathbb{C}^{p(q-1)}\right)}{\left(1 + \|w\|^2\right)^\kappa}$$

$$\leq C \int_{w \in \mathbb{C}^{p(q-1)}} \frac{dV_w\left(\mathbb{C}^{p(q-1)}\right)}{\left(1 + \|w\|^2\right)^\kappa} \leq C' \quad \text{if} \quad \kappa > p(q-1)\,.$$

Thus, we obtain that for all I in $\mathcal{I}$, $\int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^\kappa} \leq C$, where C is independent of φ.

Since $G_{p,q}(\mathbb{C})$ is compact, there exists a family $\left(V_I\right)_{I \in \mathcal{I}}$ of open sets of $G_{p,q}(\mathbb{C})$ such that V_I is relatively compact in $\pi(U_I)$ for every $I \in \mathcal{I}$, and $\bigcup_{I \in \mathcal{I}} V_I = G_{p,q}(\mathbb{C})$. There exists $M > 0$ such that $F_I \leq M$ on V_I for every $I \in \mathcal{I}$. Thus,

$$\int_{G_{p,q}(\mathbb{C})} e^{-\alpha\varphi} \leq \sum_{I \in \mathcal{I}} \int_{V_I} e^{-\alpha\varphi} \leq \sum_{I \in \mathcal{I}} M^\kappa \int_{V_I} \frac{e^{-\alpha\varphi}}{F_I^\kappa} \leq M^\kappa \sum_{I \in \mathcal{I}} \int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^\kappa} \leq C\, M^\kappa \binom{p+q}{p}.$$

We deduce that $\alpha\left(\mathcal{G}_{p,q}\right) \geq 1$.

Upper bound of $\alpha(\mathcal{G}_{p,q})$ We use here a method which can be found in [13] for the complex projective space. Let I in $\mathcal{I}$. We define $\tilde{K}$ from $M^*(p+q, p)$ to $\mathbb{P}^1(\mathbb{R})$ by the relation $\tilde{K}(M) = \left[\, |\det m_I(H)|^2, \det {}^t M \overline{M} \,\right]$. $\tilde{K}$ is invariant by the action of the structural group $G_p(\mathbb{C})$, so it induces a C^∞ map K from $G_{p,q}(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{R})$. Remark that $\psi = \log K$ is a Kähler potential on U_I for the metric $\mathcal{G}_{p,q}$.

Lemma 3.5. *There exists a decreasing sequence $\left(\varphi_n\right)_{n \geq 0}$ of admissible functions with positive maxima which converges pointwise to $-\psi$ on $\pi\left(U_I\right)$.*

Proof. We construct a decreasing sequence $\left(f_n\right)_{n \geq 0}$ of C^∞ convex functions on $\mathbb{R}_+$ satisfying the conditions $1 + f_n' > 0$, $f_n(x) = -\left(1 - 1/n\right)x$ for x in $[0, n]$ and $f_n(x) = -n$ for $x \geq 2n$. Let y be an element of $\pi(U_I)^c$ and Ω_n the set of the elements x in $\pi(U_I)$ such that $\psi(x) > 2n$. Since $F_I(y) = [0, 1]$, there exists a neighborhood V of y such that the inequality $z > e^{2n}$ holds for every point $[1, z]$ in $F_I(V)$. Thus, $V \cap \pi\left(U_I\right)$ is included in Ω_n. We have proved that $W_n = \Omega_n \cup \pi\left(U_I\right)^c$, so that W_n is an open neighborhood of $\pi\left(U_I\right)^c$. We define φ_n by $\varphi_n = f_n \circ \psi$ on $\pi\left(U_I\right)$ and $\varphi_n = -n$ on W_n. Thus, φ_n is well defined and $\varphi_n(0) = 0$. It remains to show that φ_n is admissible on $\pi\left(U_I\right)$. We have

$$\left(\mathcal{G}_{p,q} + i\,\partial\overline{\partial}\,\varphi_n\right)_{\lambda\overline{\mu}} = \partial_{\lambda\overline{\mu}}\psi + \partial_\lambda\left(f_n' \circ \psi\right)\partial_{\overline{\mu}}\psi = \left(1 + f_n' \circ \psi\right)\partial_{\lambda\overline{\mu}}\psi + f_n'' \circ \psi\, \partial_\lambda\psi\, \partial_{\overline{\mu}}\psi\,.$$

Hence, the matrix of the metric $\mathcal{G}_{p,q} + i\,\partial\overline{\partial}\,\varphi_n$ is of the form $A + T$ where A is positive definite and T has rank one and positive trace. So $A + T$ is positive definite and we get the result. □

Lemma 3.6. *Let n in $\mathbb{N}^*$ and r a positive real number. Then*

$$\int_{\|X\| \leq r} \frac{dV_X(M_n(\mathbb{C}))}{\left|\det X\right|^2} = +\infty\,.$$

Proof. We can write

$$\int_{\|X\|\leq r} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2} = \sum_{k=0}^{\infty} \int_{r/2^{k+1}\leq\|X\|\leq r/2^k} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2}.$$

We put $Y = 2^k X$, so

$$\int_{r/2^{k+1}\leq\|X\|\leq r/2^k} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2} = \int_{r/2\leq\|Y\|\leq r} \frac{dV_Y(M_n(\mathbb{C}))}{|\det Y|^2}.$$

The terms in the series are strictly positive and independent of k. The sum is therefore infinite. □

We can now prove that $\alpha(\mathcal{G}_{p,q})$ is upper bounded by 1. Suppose that $\alpha(\mathcal{G}_{p,q}) > 1$. Then there exists a positive C such that for every integer n, $\int_{\pi(U_I)} e^{-\varphi_n} \leq C$. Using Lemma 3.5 and monotonous convergence, $\int_{\pi(U_I)} F_I \leq C$. Since $\pi(U_I)^c$ has zero measure, $\int_{G_{p,q}(\mathbb{C})} F_I \leq C$. Let $\tilde{I}$ in $\mathcal{I}$ be such that $I \cap \tilde{I} = \emptyset$ (this is possible since $p \leq q$). We have $P_{\tilde{I}}\{m_I(P_{\tilde{I}})\}^{-1} = P_I$. Remark that $m_I(P_{\tilde{I}}) = m_I(Z_{\tilde{I}})$. Thus, $\det(\mathrm{Id} + {}^tZ_I\bar{Z}_I) = \det({}^tP_{\tilde{I}}\bar{P}_{\tilde{I}})\,|\det m_I(Z_{\tilde{I}})|^{-2}$. For $\|Z_{\tilde{I}}\| \leq r$, $\det({}^tP_{\tilde{I}}\bar{P}_{\tilde{I}}) \leq M$, so that $\int_{\|Z_{\tilde{I}}\|\leq r} \frac{dV_{Z_{\tilde{I}}}(M_{q,p}(\mathbb{C}))}{|\det m_I(Z_{\tilde{I}})|^2} < +\infty$. Integrating over the remaining variables $(Z_{ij})_{i\in\tilde{I}^c\cap I^c}$ yields $\int_{\|Z\|\leq r} \frac{dV_Z(M_p(\mathbb{C}))}{|\det Z|^2} < +\infty$, which is in contradiction with the result of Lemma 3.6. Thus, we obtain $\alpha(\mathcal{G}_{p,q}) \leq 1$.

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