INFINITESIMAL DEFORMATIONS OF RATIONAL SURFACE AUTOMORPHISMS

JULIEN GRIVAUX

Abstract. If X is a rational surface without nonzero holomorphic vector field and f is an automorphism of X, we study in several examples the Zariski tangent space of the local deformation space of the pair (X, f).
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Contents

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1. Introduction

Biregular automorphisms of rational surfaces with positive topological entropy present a major interest in complex dynamics (*see* the recent survey [10]) but their construction remains still a difficult problem of algebraic geometry. For an overview of this problem, we refer to [13] and to the references therein. It is a paradoxal fact that these automorphisms, although hard to construct, can occur in holomorphic families of arbitrary large dimension, as shown recently in [4]. Besides, the automorphism group of a given rational surface can carry many automorphisms of positive entropy, *see* [5] and [7] for recent results on this topic. In this paper we study deformations of families of rational surface automorphisms using deformation theory (this was initiated in [13]), and investigate a great number of examples.

Let us first describe the general setup. For the basic definitions concerning deformation theory, we refer the reader to §2. If X is a complex compact manifold, Kuranishi's theorem shows the existence of a semi-universal deformation $(\mathscr{K}, \mathcal{B}_X)$ of X, which means that any local deformation of X can be obtained by pulling back $\mathscr{K}$ by a germ of holomorphic map whose differential at the origin is unique. Kuranishi's space $\mathcal{B}_X$ is an analytic subset of $\mathrm{H}^1(X, \mathrm{T}X)$ whose Zariski tangent space at the origin is $\mathrm{H}^1(X, \mathrm{TX})$. The space $\mathcal{B}_X$ is singular if and only if there are obstructed first order deformations, that is elements in $\mathrm{H}^1(X, \mathrm{T}X)$ or equivalently deformations over the double point spec $\mathbb{C}[t]/t^2$, that cannot be lifted to deformations over a smooth base.

After the initial works of Kodaira, Spencer, Kuranishi, Horikawa, and others, deformation theory has been developed in an abstract categorical formalism mainly by Grothendieck, Artin and Schlessinger in order to cover a wide range of situations (deformations of manifolds or schemes with extra additional structure such as marked points or level structures, deformations of submanifolds, deformations of morphisms, deformations of representations, and so on) in a unified way.

In the present paper, we develop this theory for pairs (X, f) where X is a complex compact manifold and f is a biholomorphism of X. Assuming that X carries no nonzero holomorphic vector field (this guarantees that the Kuranishi family is universal), f acts naturally on the Kuranishi space $\mathcal{B}_X$. Then the restriction of the Kuranishi family $\mathscr{K}$ to the fixed locus Z_f of this action is universal for the deformation functor of pairs (X, f). Besides the Zariski tangent space of Z_f as the origin identifies with fixed vectors in $\mathrm{H}^1(X, \mathrm{T}X)$ under the action of f^*. The number of moduli of (X, f), that can be thought intuitively as the maximum number of parameters of nontrivial deformations of (X, f), is the dimension of Z_f. Knowing the action to f^* on $\mathrm{H}^1(X, \mathrm{T}X)$ we deduce a bound

$$\mathrm{m}(X, f) \leq \dim \ker (f^* - \mathrm{id}) \tag{1.1}$$

for the number of moduli of (X, f), with equality if and only if Z_f is smooth. To get a finer geometric picture, we deal separately with the different possibilities:

- If f^* has no nonzero fixed vector, then Z_f is a point, which means that there exists no nontrivial deformations of (X, f) over any base (reduced or not). In particular (X, f) is rigid.
- If $\ker (f^* - \mathrm{id})$ is nonzero, assume that we can produce a deformation $(\mathfrak{X}, f)$ of (X, f) over a smooth base (B, b) whose Kodaira-Spencer map from $\mathrm{T}_b B$ to $\ker (f^* - \mathrm{id})$ is surjective. Then Z_f is smooth, and $(\mathfrak{X}, \mathfrak{f})$ is complete (which means that pullbacks of this deformation encode all local deformations of (X, f) over any base).
- If $\ker (f^* - \mathrm{id})$ is nonzero but we don't know any specific deformation of the pair (X, f) over a smooth base, then everything can a priori happen concerning the dimension of Z_f, the bound (1.1) is the better estimate that can be obtained. In particular, nothing prevents Z_f from being the spectrum of a local artinian algebra, *i.e.* the origin is the only closed point of Z_f. In this case, (X, f) is also rigid.

The main difficulty in order to apply (1.1) in practical examples is to compute the action of $\mathrm{Aut}\,(X)$ on $\mathrm{H}^1(X, \mathrm{T}X)$, and this is far more delicate than the action of $\mathrm{Aut}(X)$ on the Neron-Severi group of X. The reason for this is that the first action is not defined for birational morphisms, whereas the second is. Our purpose in this article is to compute this action in various examples for rational surface automorphisms. The first case we deal with is the case of rational surfaces carrying a reduced effective anticanonical divisor. The most significant result we obtain is:

Theorem A. *Let X be a basic rational surface with $\mathrm{K}_X^2 < 0$ endowed with an automorphism f, and assume that $|-\mathrm{K}_X| = \{C\}$ for an irreducible curve C (such a curve is automatically f-invariant). Let P_f, Q_f and θ_f denote the characteristic polynomials of the automorphism f acting on* $\mathrm{NS}(X)$, $\mathrm{H}^1(X, \mathrm{T}X)$ *and* $\mathrm{H}^0(C, \mathrm{N}^*_{C/X})$ *respectively. Let $Q^*_f(x)$ be the unitary polynomial whose roots are the inverse of the roots of Q_f. Then:*

(i) If C is cuspidal, then

$$Q^*_f(x) = \frac{P_f(x)}{(x-1)(x-a_f^{-1})} \times \theta_f(x) \times (x - a_f^{-5}) \times (x - a_f^{-7}).$$

(ii) If C is smooth, let a_f be the action of f^* on the complex line $\mathrm{H}^0(C, \Omega^1_C)$. *Then*

$$Q^*_f(x) = \frac{P_f(x)}{x-1} \times \theta_f(x) \times (x - a_f).$$

We also prove an analogous statement if the effective anticanonical divisor is a reduced cycle of rational curves. It is worth saying that Theorem A doesn't seem to generalise easily to the case of plurianticanonical divisors. Even for Coble surfaces (that is $|-\mathrm{K}_X|$ is empty but $|-2\mathrm{K}_X|$ is not), computing the action of the automorphisms group on the vector space of infinitesimal deformations seems a non-trivial problem.

We give three applications of Theorem A. The first one deals with quadratic birational transformations of the projective plane leaving a cuspidal cubic curve globally invariant. These examples were introduced independently by McMullen [37] and Bedford-Kim [3] (*see* [14] for a unified approach). The result we get is the following (for the definitions of the orbit data and of the polynomial P_τ, we refer the reader to §3.3.1):

Theorem B. *Let (τ, n_1, n_2, n_3) be an admissible orbit data, and let μ be a root of P_τ that is not a root of unity. Let f be a birational quadratic map realizing the orbit data, fixing the cuspidal cubic $\mathscr{C}$, and having multiplier μ when restricted to the cubic $\mathscr{C}$. If X is the corresponding rational surface and g is the lift of f as an automorphism of X, then* $\mathrm{m}(X, g) \leq 3 - |\tau|$. *In particular, if τ has order three, g is rigid.*

In some sense this result is remarkable because even if the number of the blowups $n_1 + n_2 + n_3$ in these examples can be arbitrarily large, the number of moduli remains uniformly bounded (and is sometimes zero). Up to our knowledge, this yields the first known examples of rigid rational surface automorphisms with positive topological entropy. Next, we discuss examples of automorphisms fixing a smooth elliptic curve which are produced in [6] by a classical construction that appears in [21].

Theorem C. *Let p, q, r be three pairwise distinct points on smooth cubic curve $\mathscr{C}$ in $\mathbb{P}^2$, let σ_p, σ_q, σ_r be the three birational involutions fixing pointwise $\mathscr{C}$ given by the Blanc-Gizatullin construction, and let X be the corresponding blowup of $\mathbb{P}^2$ along* 15 *points on which σ_p, σ_q and σ_r lift to automorphisms. If ψ is the lift of $\sigma_p \circ \sigma_q \circ \sigma_r$ to X, then every deformation of the pair (X, ψ) is obtained by deforming the cubic $\mathscr{C}$ and the points p, q and r on it.*

Lastly, we discuss deformations of automorphisms of unnodal Halphen surfaces. The results in this section are certainly well-known to experts (*see e.g.* [11, §2]).

Using different techniques, we study a particular class of examples which admits plurianticanonical divisors: rational Kummer surfaces. Let $\mathcal{E}$ be the elliptic curve obtained by taking the quotient of the complex line $\mathbb{C}$ by the hexagonal lattice $\Lambda = \mathbb{Z}[\mathbf{j}]$, where $\mathbf{j}^3 = 1$. The group $\mathrm{GL}(2; \Lambda)$ acts linearly on the complex plane and preserves the lattice $\Lambda \times \Lambda$; therefore any element M of $\mathrm{GL}(2; \Lambda)$ induces an automorphism f_M on $\mathcal{A} = \mathcal{E} \times \mathcal{E}$ that commutes with the automorphism ϕ defined by $\phi(x, y) = (\mathbf{j}x, \mathbf{j}y)$. The automorphism f_M induces an automorphism φ_M on the desingularization X of $\mathcal{A}/\langle\phi\rangle$. The surface X is called a rational Kummer surface, it can be explicitly obtained by blowing up a very special configuration of 12 points in $\mathbb{P}^2$. By means of two different approaches, one using the Atiyah-Bott fixed point theorem and the other one using classical techniques of sheaf theory, we compute the action of $\mathrm{GL}(2; \Lambda)$ on $\mathrm{H}^1(X, \mathrm{T}X)$. This gives again new examples of rigid rational surface automorphisms with positive topological entropy.

Theorem D. *For any matrix M of infinite order in* $\mathrm{GL}(2; \mathbb{Z}[\mathbf{j}])$, *the automorphism φ_M is rigid.*

Note that the link between $\mathbb{P}^2$ and $\mathcal{A}$ via the surface X has already been very fruitful in foliation theory (*see* [12] and [34, 41]). To contribute to this dictionary, we provide an explicit description of the birational map of $\mathbb{P}^2$ induced by φ_M after blowing down twelve exceptional curves in X. The methods works in a similar manner in the case of the square lattice $\Lambda = \mathbb{Z}[\mathbf{i}]$, $(\mathbf{i}^2 = -1)$ with slightly different results.

In the last part of the paper, we address the following problem of effective algebraic geometry: if g is an explicit Cremona transformation (that is given by three homogeneous polynomials of the same degree without common factor) that lifts to an automorphism f of a rational surface X after a finite number of blowups, how to compute the action of f^* on $\mathrm{H}^1(X, \mathrm{T}X)$? Although some rational surface automorphisms have a purely geometric construction (like Coble's automorphisms for instance), others don't. A typical example is given by the automorphisms constructed in [4] and then in [13], which are given by their analytic form. Many examples introduced by physicists being also of this type, it seems necessary to develop specific methods to deal with this problem in order to give a complete picture of the subject. The strategy for solving this problem is long and not particularly easy to grasp because we must develop some specific machinery in order to construct explicit bases of $\mathrm{H}^1(X, \mathrm{T}X)$. To help the reader understand how the algorithm works, we carry out completely the computation in one specific example, which is the one constructed in [13, Thm 3.5]; it is an automorphism obtained by blowing up 15 successive points of $\mathbb{P}^2$, which correspond to three infinitely near points of length five. This is one the most simple possible example we know with iterated blowups.

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2. Preliminary results in deformation theory

For general background on the theory of deformations of complex compact manifolds, we refer to the references [32],[38] and [44].

2.1. **Background.**

2.1.1. *Deformations and Kuranishi space.*

– If X is a smooth complex compact manifold, a (local) deformation [1] of X is an equivalence class of cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \{b\} & \longrightarrow & B \end{array}$$

where (B, b) is a pointed complex space, $\mathfrak{X}$ is a complex space, and π is is a flat and proper holomorphic morphism. In other words, the deformation is given by the morphism $(\mathfrak{X}, \pi)$ over the marked basis (B, b) together with a specific identification between the central fiber $\mathfrak{X}_b$ and X.

1. For some authors, what we call *deformation* is called *marked deformation*, because we specify the isomorphism between the central fiber and X.

– Every complex manifold defines a contravariant deformation functor

$$\mathrm{Def}_X\colon \{\text{germs of marked complex spaces}\} \longrightarrow \text{Sets}$$

given by

$$\mathrm{Def}_X(B,b) = \{\,\text{Deformations of } X \text{ over } (B,b)\,\}\,/\,\text{isomorphism}$$

– An infinitesimal deformation of X is a deformation of X over $\operatorname{spec} A$ where A is a local artinian algebra over the complex numbers.

– A first-order deformation of X is a deformation of X over $\operatorname{spec}\mathbb{C}[t]/t^2$. Concretely, if X is given in coordinate charts by a cocycle $\varphi_{\alpha\beta}$ of transition functions, an infinitesimal deformation of X is a family $f_{\alpha\beta}$ of holomorphic maps such that $(\varphi_{\alpha\beta} + \epsilon f_{\alpha\beta})$ is a cocycle, where ϵ is a formal parameter satisfying $\epsilon^2 = 0$.

– There is a natural isomorphism between $\mathrm{Def}_X(\operatorname{spec}\mathbb{C}[t]/t^2)$, that is isomorphism classes of first-order deformations of X, and $\mathrm{H}^1(X, \mathrm{TX})$. Using the previous notation, the class of the deformation given by the cocycle $\{\varphi_{\alpha\beta} + \epsilon f_{\alpha\beta}\}$ in $\mathrm{H}^1(X, \mathrm{T}X)$ is represented by the 1-Čech cocycle of holomorphic vector fields $Z_{\alpha\beta} = \sum_\gamma f^\gamma_{\alpha\beta}\frac{\partial}{\partial z_\gamma}$ (*see* [44, Proposition 1.2.9]).

– If X has no nonzero holomorphic vector fields on X and if $\mathfrak{X}$ is any infinitesimal deformation of X, then the group of automorphisms of $\mathfrak{X}$ is trivial [44, Corollary 2.6.3]. The same results holds for arbitrary deformations of X [2].

– The Kodaira-Spencer map of $\mathfrak{X}$ is a linear map $\mathrm{KS}(\mathfrak{X})$ from T_bB to $\mathrm{H}^1(X, \mathrm{T}X)$. For any vector v in T_bB, the element $\mathrm{KS}_b(\mathfrak{X})(v)$ is exactly the class of the first-order deformation $\mathfrak{X} \times_B \operatorname{spec}\mathbb{C}[t]/t^2$ of X, the map from $\operatorname{spec}\mathbb{C}[t]/t^2$ to B being given by the tangent vector v.

– A deformation of the manifold X is called universal (resp. semi-universal [3], resp. complete) if any deformation of X is the pullback of this deformation under a holomorphic map that is unique (resp. whose differential at the origin is unique, resp. without any further conditions).

– A deformation over a smooth base is complete if and only if its Kodaira-Spencer map is surjective (Kodaira's completeness theorem [33], *see* [32, Theorem 6.1]). Besides, the Kodaira-Spencer map of a semi-universal deformation is (almost by definition) an isomorphism.

– If the manifold X has no nonzero holomorphic vector field, then any semi-universal deformation of X is in fact universal (*see* [44, Cor. 2.6.3]).

– A complex manifold is rigid if all fibers of a deformation of X over a smooth base are biholomorphic to X. Thanks to a theorem of Grauert and Fischer [16], this is equivalent of saying that every deformation of X over a smooth base is locally trivial.

– Any complex manifold admits a semi-universal deformation (Kuranishi's theorem); its base $\mathcal{B}_X$ is called the Kuranishi space and the deformation $\mathscr{K}$ is called the Kuranishi family. the space $\mathcal{B}_X$ is an analytic subset of $\mathrm{H}^1(X, \mathrm{T}X)$ whose Zariski tangent space at the origin is $\mathrm{H}^1(X, \mathrm{TX})$.

2. This result is true in the analytic category, but becomes false in the algebraic category: isotrivial deformations are locally trivial for the analytic topology, but not for the Zariski topology (*see* [44, §2.6.2]).

3. Some authors use the terminology *versal*.

– Another way of stating Kuranishi's theorem is that the deformation functor Def_X attached to X is quasi-representable (*see* [48, Def. 1.2]) by the Kuranishi family $(\mathscr{B}_X, \mathscr{K})$. If X carries no nonzero holomorphic vector field, then Def_X is represented by $\mathscr{B}_X$ (*i.e.* the Kuranishi family is universal).

– The space $\mathscr{B}_X$ is in general neither reduced nor irreducible (*see* [43]). If $\mathscr{B}_X$ is smooth, then we say that X is unobstructed. This is in particular the case when the cohomology group $\mathrm{H}^2(X, \mathrm{T}X)$ vanishes, thanks to Kodaira's existence theorem [32, Thm. 5.6 pp. 270].

– The number of moduli of X, denoted by $\mathrm{m}(X)$, is the maximum of the dimensions of the irreducible components of (X, f). The number $\mathrm{m}(X)$ is in $[\![0, \mathrm{h}^1(X, \mathrm{T}X)]\!]$. Besides:

$$\mathrm{m}(X) = 0 \Leftrightarrow \mathscr{B}_X^{\mathrm{red}} = \{0\} \Leftrightarrow X \text{ is rigid.}$$
$$\mathrm{m}(X) = \mathrm{h}^1(X, \mathrm{T}X) \Leftrightarrow \mathscr{B}_X \text{ is smooth} \Leftrightarrow X \text{ is unobstructed.}$$

2.1.2. *Families of complex manifolds.* In this section, we recall briefly some of the material formerely introduced in [13, §5] concerning the generic number of parameters of a family of complex manifolds, and relate it to the number of moduli.

– A family of complex manifolds is a triplet $(\mathfrak{X}, \pi, B)$ where $\pi\colon \mathfrak{X} \to B$ is a proper submersion between smooth complex manifolds.

– For any point b in B, the family $\mathfrak{X}$ induces a deformation of $\mathfrak{X}_b$. We denote by $\mathrm{KS}_b(\mathfrak{X})$ the corresponding Kodaira-Spencer map, which is a linear map from $\mathrm{T}_b B$ to $\mathrm{H}^1(\mathfrak{X}_b, \mathrm{T}\mathfrak{X}_b)$.

– The function $b \to \mathrm{rank}\,\{\mathrm{KS}_b(\mathfrak{X})\}$ is generically constant on B. If $\mathfrak{X}$ is an algebraic family, this is proved in [13, Proposition 5.5]. For arbitrary deformations, it follows from [17, Satz 7.7(1)] that the function $b \to \mathrm{h}^1(\mathfrak{X}_t, \mathrm{T}\mathfrak{X}_t)$ is constructible, in particular it is generically constant. Then the argument of *loc. cit.* applies.

– The generic number of parameters of a family $(\mathfrak{X}, \pi, B)$, denoted by $\mathfrak{m}(\mathfrak{X})$, is the generic rank of the function $b \to \mathrm{rank}\,\{\mathrm{KS}_b(\mathfrak{X})\}$.

– Let X be a complex compact manifold without nonzero holomorphic vector field. By the semi-continuity theorem [32, Thm. 7.8], the fibers of the Kuranishi family have no nonzero vector fields either. Then it follows from [38, Corollary 1] that the Kuranishi family is semi-universal (and even universal) at any point of the base $\mathscr{B}_X$. As a corollary we get the following important result:

Proposition 2.1. *Let X be a complex manifold without nonzero holomorphic vector field. Then the number of moduli* $\mathrm{m}(X)$ *of X is the supremum of* $\mathfrak{m}(\mathfrak{X})$ *where $\mathfrak{X}$ runs through deformations over X over a smooth base.*

Proof. Let $\mathfrak{X}$ be a deformation of X over a smooth base B. We can write $\mathfrak{X}$ as $\varphi^* \mathscr{K}$ where $\varphi\colon B \to \mathscr{B}_X$ is a germ of holomorphic map. Since $\mathscr{K}$ is semi-universal at all points of the base, this implies that $\mathfrak{m}(\mathfrak{X})$ is the generic rank of φ. Since the base of $\mathfrak{X}$ is smooth, we get the inequality $\mathfrak{m}(\mathfrak{X}) \leq \mathrm{m}(X)$. To prove the equality, let Z be an irreducible component of maximal dimension of $\mathscr{B}_X$ and take for φ the composition $B \to Z^{\mathrm{red}} \to Z$ where Z^{red} is the reduction of Z, and the first morphism is a resolution of singularities. Then $\mathfrak{m}(\varphi^* \mathscr{K}) = \dim Z^{\mathrm{red}} = \mathrm{m}(X)$.

□

2.2. **Deformations of automorphisms.**

2.2.1. *Setting.* Let X be a complex compact manifold without nonzero holomorphic vector field, and let f be a biholomorphism of X. A deformation of the pair (X, f) is an equivalence class of cartesian diagram

$$\begin{array}{ccc} (X, f) & \longrightarrow & (\mathfrak{X}, \mathfrak{f}) \\ \downarrow & & \downarrow \pi \\ \{b\} & \longrightarrow & B \end{array}$$

where B is a germ of marked complex space, π is flat and proper, $\mathfrak{f}$ is an biholormorphism of $\mathfrak{X}$ commuting with π, and the top horizontal arrow commutes with the automorphisms. There is also a deformation functor $\mathrm{Def}_{(X,f)}$ from germs of marked complex spaces to sets encoding the deformations of (X, f) modulo isomorphisms.

Lemma 2.2. *For any marked base* (B, b)*, the natural map* $\mathrm{Def}_{(X,f)}(B, b) \to \mathrm{Def}_X(B, b)$ *is injective.*

Proof. We must prove that if $(\mathfrak{X}, \mathfrak{f})$ is in $\mathrm{Def}_X(B, b)$, then $\mathfrak{f}$ is uniquely determined by $\mathfrak{X}$. Taking two possible biholomorphisms $\mathfrak{f}$ and $\mathfrak{f}'$, $\mathfrak{f}' \circ \mathfrak{f}^{-1}$ is an automorphism of $\mathfrak{X}$, so it is the identity morphism (since we have assumed that X has no nonzero holomorphic vector field, *see* §2.1.1). □

If $(\mathfrak{X}, \mathfrak{f})$ is a deformation of a pair (X, f), then for any point b in B we have $\mathfrak{f}_b^* \circ \mathrm{KS}_b = \mathrm{KS}_b$, so that the image of KS_b is contained in $\ker(\mathfrak{f}_b^* - \mathrm{id})$. It is therefore natural to define the Kodaira-Spencer map of the pair (X, f) at any point b in B as the unique map

$$\mathrm{KS}_b(\mathfrak{X}, \mathfrak{f})\colon \mathrm{T}_b B \longrightarrow \ker(f^* - \mathrm{id})$$

such that the composition $\mathrm{T}_b B \xrightarrow{\mathrm{KS}_b(\mathfrak{X},\mathfrak{f})} \ker(f^* - \mathrm{id}) \hookrightarrow \mathrm{H}^1(X, \mathrm{T}X)$ is $\mathrm{KS}_b(\mathfrak{X})$.

Definition 2.3. We say that (X, f) is *rigid* if for any deformation $(\mathfrak{X}, \mathfrak{f})$ of (X, f) over a smooth base B, for any b in B, $(\mathfrak{X}_b, \mathfrak{f}_b)$ is biholomorphic to (X, f).

Remark that (X, f) is rigid if and only if any deformation of (X, f) over a smooth base is locally trivial. Indeed, if $(\mathfrak{X}, \mathfrak{f})$ is such a deformation, all fibers of $\mathfrak{X}$ are biholomorphic so using the Fischer-Grauert theorem [16], $\mathfrak{X}$ is locally trivial. Lemma 2.2 implies that $(\mathfrak{X}, \mathfrak{f})$ is also locally trivial.

2.2.2. *The invariant locus.* In this section, we fix a complex compact manifold X without nonzero holomorphic vector field. For any deformation $\mathfrak{X}$ of X, we have a new deformation $\mathfrak{X}^f$ of X obtained from $\mathfrak{X}$ by pre-composing the deformation $\mathfrak{X}$ with f, that is by considering the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & X & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow & & \downarrow \\ \{b\} & = & \{b\} & \longrightarrow & B \end{array}$$

Remark that if $(\mathfrak{X}, \mathfrak{f})$ is in $\mathrm{Def}_{(X,f)}$, then $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}^f$ is an isomorphism of deformations. Since $\mathscr{K}$ is universal, there exists a unique germ of biholomorphism φ_f fixing the marked point b such that $\mathscr{K}^f \simeq \varphi_f^* \mathscr{K}$ as deformations of X[4]. This means that there exists a biholomorphism $\widehat{f}$ of $\mathscr{K}$ such

4. For semi-universal deformations φ_f is not unique anymore, and this can be the source of many problems. *See* [42] and [45] for further details.

that the diagrams

$$\begin{array}{ccc} \mathscr{K} & \xrightarrow{\widehat{f}} & \mathscr{K} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\varphi_f} & B \end{array} \qquad\qquad \begin{array}{ccc} \mathscr{K} & \xrightarrow{\widehat{f}} & \mathscr{K} \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & X \end{array}$$

commute.

Definition 2.4. Given a pair (X, f), if $\mathcal{B}_X$ is the Kuranishi space of X, we define the f-invariant locus of X as the subscheme Z_f of $\mathcal{B}_X$ defined as the (schematic) pullback of the diagonal of $\mathcal{B}_X$ under the map $(\mathrm{id}, \varphi_f)\colon \mathcal{B}_X \to \mathcal{B}_X \times \mathcal{B}_X$. We also define the number of moduli $\mathrm{m}(X, f)$ of (X, f) as the maximum of the dimensions of the irreducible components of Z_f.

If we identify $\mathrm{T}_b\mathcal{B}_X$ with $\mathrm{H}^1(X, \mathrm{T}X)$, then $\mathrm{T}_b Z_f$ identifies with $\ker(f^* - \mathrm{id})$. Taking the pullback of the above left diagram under the inclusion $Z_f \hookrightarrow \mathcal{B}_X$, we get a diagram

$$\begin{array}{ccc} \mathscr{K}_{|Z_f} & \longrightarrow & \mathscr{K}_{|Z_f} \\ & \searrow \quad \swarrow & \\ & Z_f & \end{array}$$

Hence $(\mathscr{K}_{|Z_f}, \widehat{f})$ is an element of $\mathrm{Def}_{(X,f)}$.

Proposition 2.5. *Let X be a complex compact manifold with no holomorphic vector field, let f be a biholomorphism of X, let $\mathcal{B}_X$ the Kuranishi space of X, and let Z_f be the f-invariant locus. Then the element $(\mathscr{K}_{|Z_f}, \widehat{f})$ is universal for the functor* $\mathrm{Def}_{(X,f)}$.

Proof. Let $(\mathfrak{X}, \mathfrak{f})$ be a deformation of (X, f) over a germ of marked complex space (M, m). Since the deformation $\mathscr{K}$ is universal, there exists a unique germ $\psi\colon (M, m) \to (\mathcal{B}_X, b)$ of holomorphic map such that $\mathfrak{X} \simeq \psi^*\mathscr{K}$. Now $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}^f$ is an isomorphism of deformations. This means that $\psi^*\mathscr{K}$ and $(\varphi_f \circ \psi)^*\mathscr{K}$ are isomorphic. Hence that $\varphi_f \circ \psi = \psi$ (as $\mathscr{K}$ is universal), so that the map ψ factors through Z_f. We conclude that the deformations $\mathfrak{X}$ and $\psi^*\mathscr{K}_{|Z_f}$ are isomorphic. Thanks to Lemma 2.2, $(\mathfrak{X}, \mathfrak{f})$ is isomorphic to $(\psi^*\mathscr{K}_{|Z_f}, \widehat{f})$. □

Remarks 2.6.

(i) As usual in deformation theory, the singularities of the moduli space (here Z_f) correspond to obstructed deformations. An extreme case could be the following: take local coordinates near b_0, and assume [5] that in these coordinates, φ_f is given by

$$\varphi_f(z_1, \ldots, z_n) = (z_1 + z_1^2, z_2 + z_2^2, \ldots, z_n + z_n^2).$$

Then Z_f is a non-reduced scheme supported at the point $(0, \ldots, 0)$, and its Zariski tangent space is $\mathbb{C}^n$. The action of f^* on $\mathrm{H}^1(X, \mathrm{T}X)$ is trivial, but all infinitesimal deformations are obstructed.

(ii) If f is linearizable (*e.g.* f is of finite order), then Z_f is smooth.

5. We don't have any example of such f, so this remark is purely speculative.

The following theorem is a generalisation of Kodaira's completeness theorem [32, Thm. 6.1] for deformations of automorphisms. Its proof is a direct consequence of a general result on complete families for semi-universal deformation functors, due to Wavrik [48, Thm 1.8].

Theorem 2.7 (Completeness theorem)**.** *Let X be a complex compact manifold without nonzero holomorphic vector field, and let f be an automorphism of X. Consider a deformation $(\mathfrak{X}, \mathfrak{f})$ of the pair (X, f) over a smooth base (B, b), and assume that $\mathrm{KS}_b(\mathfrak{X}, \mathfrak{f})$ is surjective. Then Z_f is smooth, and $(\mathfrak{X}, \mathfrak{f})$ is complete at b.*

Proof. By Proposition 2.5, we can write $\mathfrak{X} = \psi^* \mathscr{K}$ where $\psi \colon B \to Z_f$ is holomorphic. Since the differential $\mathrm{d}\psi_b \colon \mathrm{T}_b B \to \mathrm{T}_{\psi(b)} Z_f$ is surjective, Z_f is smooth and ψ is a submersion. Hence ψ admits locally a right inverse $\chi \colon Z_f \to B$. This gives $\chi^* \mathfrak{X} = \mathscr{K}_{|Z_f}$, and we conclude using Lemma 2.2. □

Lastly, we provide a concrete interpretation of the number of moduli of a pair (X, f), whose proof is entirely similar to the proof of Proposition 2.1.

Proposition 2.8. *Let X be a complex compact manifold without nonzero holomorphic vector field, and let f be a biholomorphism of X. Then $\mathrm{m}(X, f)$ is the supremum of $\mathfrak{m}(\mathfrak{X})$ where $\mathfrak{X}$ runs through deformations of X over a smooth base such that f extends to an automorphism of the deformation $\mathfrak{X}$. Besides, $\mathrm{m}(X, f) \leq \dim \ker (f^* - \mathrm{id})$, and*

$$\mathrm{m}(X, f) = 0 \Leftrightarrow Z_f^{\mathrm{red}} = \{0\} \Leftrightarrow (X, f) \text{ is rigid.}$$
$$\mathrm{m}(X, f) = \dim \ker (f^* - \mathrm{id}) \Leftrightarrow Z_f \text{ is smooth} \Leftrightarrow (X, f) \text{ is unobstructed.}$$

3. Automorphisms of anticanonical basic rational surfaces

3.1. General results on rational surfaces.

3.1.1. *Basic rational surfaces.*

– Let X be a smooth projective surface, let $\mathrm{Aut}\,(X)$ be its group of biholomorphisms, let $\mathrm{NS}(X)$ its Neron-Severi group, and let $\varepsilon_X \colon \mathrm{Aut}\,(X) \to \mathrm{GL}\{\mathrm{NS}(X)\}$ be the natural representation given by $f \to f^*$. The first dynamical degree $\lambda_1(f)$ of an element f of $\mathrm{Aut}(X)$ is the spectral radius of $\varepsilon_X(f)$.

– Thanks to [24] and [49], the topological entropy of f is $\log\, \lambda_1(f)$. Although we won't use it in this paper, the notion of first dynamical degree can be defined for birational maps between projective complex surfaces and is invariant by birational conjugacy.

– Let X be a rational surface. It is well-known that X is isomorphic to a finite blowup of $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$. By definition, a basic rational surface is a finite blowup of $\mathbb{P}^2$.

– For any projective surface X, the Hirzebruch-Riemann-Roch theorem [30, Thm. 5.1.1] and the Gauss-Bonnet theorem [22, p. 416] give

$$\mathrm{h}^0(X, \mathrm{T}X) - \mathrm{h}^1(X, \mathrm{T}X) + \mathrm{h}^2(X, \mathrm{T}X) = \frac{7 c_1(X)^2 - 5 c_2(X)}{6} = \frac{7\, \mathrm{K}_X^2 - 5 \chi(X)}{6}.$$

If X is a basic rational surface obtained by successively blowing up the projective plane N times, we have $\mathrm{h}^2(X, \mathrm{T}X) = 0$ (since the quantity $\mathrm{h}^2(X, \mathrm{T}X)$ is a birational invariant thanks to Serre duality, and vanishes for $X = \mathbb{P}^2$). Besides, $\mathrm{K}_X^2 = 9 - N$, and $\chi(X) = 3 + N$ so that assuming that X has no nonzero holomorphic vector field, $\mathrm{h}^1(X, \mathrm{T}X) = 2N - 8$.

– Thanks to a result of Nagata [39, Th. 5], if X is a rational surface and if $\mathrm{Im}(\varepsilon_X)$ is infinite, then X is basic. Besides, by result of Harbourne [25, Cor. 4.1], for any rational surface X, $\mathrm{Ker}(\varepsilon_X)$ and $\mathrm{Im}(\varepsilon_X)$ cannot be both infinite. Combining these two results, if X is a rational surface endowed with an automorphism whose action on the Picard group is of infinite order (*e.g.* with positive topological entropy), then X is basic and has no nonzero holomorphic vector field.

– Let us now give a brief description of the Kuranishi space attached to a basic rational surface without nonzero holomorphic vector field. Note that if X is such a surface, $\mathrm{H}^2(X, \mathrm{T}X)$ vanishes so X is unobstructed. For any integer N, let S_N be the Fulton-MacPherson configuration space of ordered (possibly infinitely near) N-uplets of points in the projective plane (*see* [19]), and let $\mathfrak{X}_N$ be the corresponding family of rational surfaces given by $(\mathfrak{X}_N)_{\widehat{\xi}} = \mathrm{Bl}_{\widehat{\xi}}\mathbb{P}^2$. The group $\mathrm{PGL}(3;\mathbb{C})$ acts naturally on $(\mathfrak{X}_N, \pi, S_N)$ via its standard action on $\mathbb{P}^2$. We denote by $S_N^{\dagger}$ the Zariski open subset of S_N where the action is free. It parametrizes rational surfaces without nonzero holomorphic vector field. For any $\widehat{\xi}$ in S_N, let $O_{\widehat{\xi}}$ be the $\mathrm{PGL}(3;\mathbb{C})$-orbit passing through $\widehat{\xi}$. We have the following result [13, Thm. 5.1]:

Proposition 3.1. *The family $\mathfrak{X}_N$ induces a complete deformation at every point of $S_N^{\dagger}$, and for any point $\widehat{\xi}$ of S_N we have an exact sequence*

$$0 \longrightarrow \mathrm{T}_{\widehat{\xi}} O_{\widehat{\xi}} \longrightarrow \mathrm{T}_{\widehat{\xi}} S_N \xrightarrow{\mathrm{KS}_{\widehat{\xi}}(\mathfrak{X}_N)} \mathrm{H}^1(X, \mathrm{T}X) \longrightarrow 0.$$

– As a corollary, if $\widehat{\xi}$ is in $\mathrm{S}_N^{\dagger}$, the restriction of X_N to any smooth submanifold of codimension 8 passing through $\widehat{\xi}$ and transverse to $O_{\widehat{\xi}}$ yields a deformation whose Kodaira-Spencer map is an isomorphism at every point, it is the Kuranishi space of X.

– Proposition 3.1 allows to compute the generic number of parameters of families of basic rational surfaces (*see* [13, Thm. 5.8] for a precise statement). As a particular case, we have the following result:

Proposition 3.2. *Let $N \geq 4$ be an integer, and $\mathcal{V}$ be a connected submanifold of $S_N^{\dagger}$ which is stable under the action of* $\mathrm{PGL}(3;\mathbb{C})$. *Then the Kodaira-Spencer of the deformation $\mathfrak{X}_{N|\mathcal{V}}$ has everywhere rank* $\dim \mathcal{V} - 8$.

3.1.2. *Anticanonical surfaces.* Anticanonical surfaces are surfaces whose anticanonical class is effective. These surfaces play a crucial role in the theory of rational surfaces (*see e.g.* [35] and [26]). Let us give at first a classification of possible reduced [6] anticanonical divisors on a projective surface. We start by a simple lemma:

Lemma 3.3. *Let Y is a smooth surface and $\pi\colon X \to Y$ is a point blowup and D is an effective anticanonical divisor on X. Then $\pi_*(D)$ is an effective anticanonical divisor on Y, and $D = \pi^*\pi_*(D) - E$. Besides, D is connected if and only if $\pi(D)$ is connected.*

Proof. Let E be the exceptional divisor, $U = X \setminus E$ and $p = \pi(E)$. Then $D_{|U}$ is an anticanonical divisor on U. It follows that K_Y and $\pi_* D$ are isomorphic on U, and therefore on Y thanks to Hartog's theorem. Hence $\pi_* D$ is anticanonical.

6. If D is reducible but non reduced, the situation gets more complicated, even if we know that the irreducible components are still smooth rational curves.

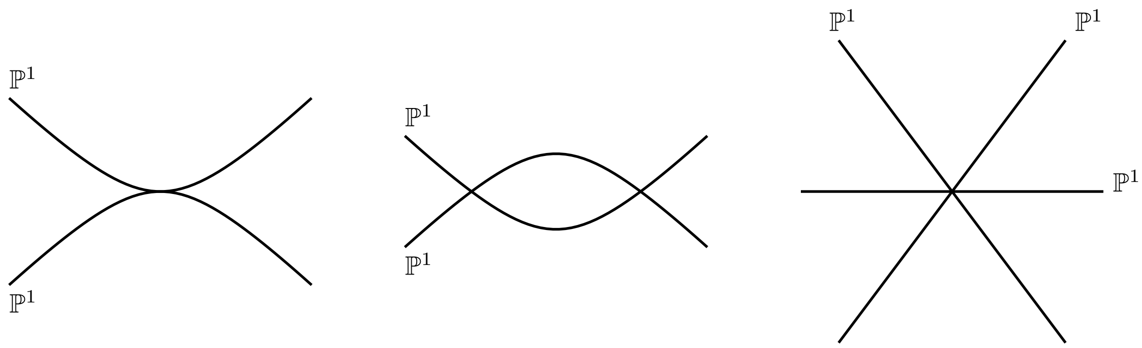
The divisor $\pi^*\pi_* D - D$ is equal to mE for some m in $\mathbb{Z}$. Besides, since $\pi^*\mathrm{K}_Y = \mathrm{K}_X + E$, the divisor $\pi^*\pi_* D - D$ is linearly equivalent to E. Hence $(m-1)E$ is linearly equivalent to zero, which forces $m = 1$ since $E^2 \neq 0$.

For the last point, let us write $D = \overline{Z} + mE$ where Z is an effective divisor in Y, $\overline{Z}$ is the strict transform of Z in X and m is in $\mathbb{N}$. Note that p belongs to Z, otherwise $\pi^*\pi_* D - E = \pi^* Z - E = \overline{Z} - E$ is not equal to D. Hence, if $m \geq 1$, $\overline{Z}$ and E meet. This implies the required result. □

Corollary 3.4. *If X is a basic rational surface, any effective anticanonical divisor is connected.*

It is possible to classify connected reduced anticanonical divisors (*see* [15, Th. 4.2] in a slightly different context):

Proposition 3.5. *Let X be a smooth projective surface, and let D be a reduced and connected effective divisor representing the class $-K_X$. Then D is either an irreducible reduced curve of arithmetic genus 1, or a cycle of smooth rational curves, or one of the three exceptional configurations shown in the picture below*:



Proof. Let us write $D = \sum_{i=1}^{k} D_i$. For any i the arithmetic genus of D_i is

$$g(D_i) = 1 + \frac{1}{2} D_i\,(D_i + K_X) = 1 - \frac{1}{2} \sum_{j \neq i} D_i.D_j$$

so that $g(D_i) \leq 1$.

(i) Assume that for some i, $g(D_i) = 1$. Then D_i is a connected component of D, so that $D = D_i$ and D_i is a reduced curve of arithmetic genus 1.

(ii) Assume that for all i, $g(D_i) = 0$. Then all the D_i's are smooth rational curves and we have $D_i . \sum_{j \neq i} D_j = 2$. If for some indices i and j we have $D_i.D_j = 2$ then $D_i + D_j$ is a connected component of D and we are in the first two exceptional configurations. Otherwise, all D_i's intersect transversally. If for some indices i, j and k the intersection $D_i \cap D_j \cap D_k$ is nonempty then $D_i + D_j + D_k$ is again a connected component of D and we are in the third exceptional configuration. The last remaining possibility is that for any i, there exists exactly two components D_j and D_k intersecting D_i, and satisfying $D_j \cap D_k = \varnothing$. Hence D is a cycle of smooth rational curves.

□

Remark 3.6. All possible configurations of reduced effective anticanonical divisors can occur on rational surfaces carrying an automorphism of positive entropy, except possibly the case of an irreducible nodal curve of arithmetic genus one (even in the case of asymptotically stable birational maps, this case is not ruled out in [15, Th. 4.2]). To see this, we list all the possible cases. Let D denote a reduced anticanonical divisor. The construction of Blanc and Gizatullin [6] provides examples of rational surfaces carrying a smooth anticanonical elliptic curve (*see also* [14, Ex. 3.3] for an example producing a quadratic transformation fixing the square torus and lifting to an automorphism of positive entropy). This construction will be given in details in §3.3.2. All other configurations except the irreducible nodal curve can occur using quadratic transformations in $\mathbb{P}^2$. The case of a cuspidal elliptic curve goes back to McMullen [37] (*see* [14, Thm 3.5 & Thm 3.6]), we will also recall it in §3.3.1. For the first exceptional configuration, this is said to be possible in the last paragraph before [14, §4.1], although the precise result is not stated. For the second configuration, *see* [14, Thm. 4.5]. For the third configuration, *see* [14, Thm. 4.4].

We end this section by discussing the existence of holomorphic vector fields on anticanonical surfaces.

Lemma 3.7. *Let X be a basic rational surface with $\mathrm{K}_X^2 < 0$ admitting an irreducible and reduced anticanonical curve. Then X admits no nonzero holomorphic vector field.*

Proof. Let G be the connected component of the identity in the automorphism group of X, and let C be an irreducible and reduced curve in $|-\mathrm{K}_X|$. Since $\mathrm{K}_X^2 < 0$, C is fixed by G, as well as any (-1)-curve on X. Therefore, if $\mathcal{S}$ is the finite set of (possibly infinitely near) points in $\mathbb{P}^2$ that are blown up to get X, $\mathrm{Aut}^0(X)$ identifies with the connected component of the identity of the algebraic subgroup of $\mathrm{PGL}(3;\mathbb{C})$ consisting of linear transformations fixing $\mathcal{S}$ pointwise. If $\pi\colon X \to \mathbb{P}^2$ is the projection map, then $\pi(C)$ is an irreducible reduced cubic, and $\mathcal{S}$ consists of at least ten points supported in the smooth locus of $\pi(C)$. But the subgroup of $\mathrm{PGL}(3;\mathbb{C})$ fixing an irreducible reduced cubic in $\mathbb{P}^2$ and a smooth point on it is finite. This proves that $G = \{\mathrm{id}\}$ so that X has no nonzero holomorphic vector field. □

3.2. **Action of the automorphisms group on first order deformations.**

3.2.1. *Elliptic curves.* – We start with some preliminary results on equivariant bundles on elliptic curves, which are probably classical although we couldn't find them in the literature. For any divisor D on a curve C, we denote by $[D]$ the holomorphic line bundle $\mathcal{O}_C(D)$.

– If C is an irreducible reduced curve of arithmetic genus 1, let C^{reg} be the smooth locus of C. Given any base point P_0 in C^{reg}, the map from C^{reg} to $\mathrm{Pic}^0(C)$ given by $P \to [P] - [P_0]$ is an isomorphism. Hence C^{reg} is naturally endowed with the structure of an algebraic group having P_0 as origin. This group is of the form $\mathbb{C}/\Gamma$ where Γ is a discrete subgroup of $\mathbb{C}$ of rank 0, 1 or 2 depending on the three cases C cuspidal, C nodal or C smooth.

– If we endow C^{reg} with its algebraic group structure, the map from $C^{\mathrm{reg}} \oplus \mathbb{Z}$ to $\mathrm{Pic}\,(C)$ given by

$$(P, n) \to [P] + (n-1)[P_0]$$

is a group isomorphism.

– Any biholomorphism φ of C preserves C^{reg} and lifts to an affine map $z \to az + b$ of $\mathbb{C}$, where the multiplier a satisfies $a\Gamma = \Gamma$ and is independent of the lift. A more intrinsic way of defining the multiplier is as follows: if ω_C is the dualizing sheaf of C, $\mathrm{H}^0(C, \omega_C)$ is a complex line. The action φ^* on $\mathrm{H}^0(C, \omega_C)$ is the multiplication by a. The number b is called the translation factor; it is well defined modulo Γ but it depends of the choice of the origin.

Proposition 3.8. *Let C be an irreducible reduced curve of arithmetic genus* 1*, let $\mathcal{L}$ be a holomorphic line bundle on C of positive degree n, let $\mathcal{G}_{\mathcal{L}}$ be the stabilizer of $\mathcal{L}$ in* $\mathrm{Aut}\,(C)$*, and let χ be the character of $\mathcal{G}_{\mathcal{L}}$ given by the multiplier.*

(i) Assume that C is smooth.

- ***Similitudes**. For any morphism φ in $\mathcal{G}_{\mathcal{L}}$ that is not a translation, let P be any fixed point of φ. Then the line bundle $\mathcal{L}$ can be endowed with an action of the cyclic group $\langle\varphi\rangle$ generated by φ such that the action on the fiber $\mathcal{L}_P$ is trivial, and that the corresponding representation on* $\mathrm{H}^0(C, \mathcal{L})$ *is :*

$$\begin{cases} \displaystyle\bigoplus_{0\le k\le n,\ k\neq n-1} \chi^k & \textit{if} \quad \mathcal{L} \sim n[P] \\ \displaystyle\bigoplus_{0\le k\le n-1} \chi^k & \textit{otherwise.} \end{cases}$$

- ***Translations**. The intersection of $\mathcal{G}_{\mathcal{L}}$ with the group of translations by elements of C is isomorphic to the group T_n of n-torsion points of C. If b is a primitive n-torsion point and $\langle b\rangle$ is the cyclic group generated by b in H_n, then $\mathcal{L}$ can be endowed with a $\langle b\rangle$-action whose representation of $\langle b\rangle$ on* $\mathrm{H}^0(C, \mathcal{L})$ *is $\bigoplus_{0\le k\le n-1} \nu^k$ where ν is any primitive character of $\langle b\rangle$.*

(ii) If C is cuspidal, the morphism $\chi\colon \mathcal{G}_{\mathcal{L}} \to \mathbb{C}^\times$ is an isomorphism, and there is a unique global fixed point P on C^{reg} under the action of $\mathbb{C}^\times$. Besides, $\mathcal{L}$ can be endowed with an action of $\mathbb{C}^\times$ which is trivial on $\mathcal{L}_P$, and such that the corresponding representation of $\mathbb{C}^\times$ on $\mathrm{H}^0(C, \mathcal{L})$ *is $\bigoplus_{0\le k\le n,\ k\neq n-1} \chi^k$.*

Proof. Let φ be an automorphism of C that lifts to the map $z \to az + b$. If $\mathcal{L} \sim (n-1)[0] + [w]$, we have $\varphi_*\mathcal{L} \sim (n-1)[b] + [aw + b] \sim (n-1)[0] + [aw + nb]$ so that the isomorphism class of $\mathcal{L}$ is fixed by φ if and only if $aw + nb = w \mod \Gamma$.

If $\mathrm{rank}\,(\Gamma) = 2$, pick such a φ with $a \neq 1$ (so that φ is not a translation), and choose for the origin of C a fixed point P of φ. Then the translation factor of φ vanishes so that $aw = w \mod \Gamma$. Remark now that the divisor $D = (n-1)\,0 + w$ is invariant by φ and $\mathcal{L}$ is isomorphic to $[D]$. There is therefore a natural $\langle\varphi\rangle$-action on the line bundle $\mathcal{L}$, and the corresponding action on $\mathrm{H}^0(C, \mathcal{L})$ is given by precomposing with φ. For any meromorphic function u in $\mathrm{H}^0(C, \mathcal{L})$, let $\alpha_0(u), \dots, \alpha_n(u)$ be the Laurent coefficients of u near 0:

$$u(z) = \sum_{i=0}^{n} \alpha_i(u) z^{-i} + \mathrm{O}(z)$$

There is a natural isomorphism between $\mathrm{H}^0(C, \mathcal{L})$ and $\mathbb{C}^n$ obtained as follows:

$$\begin{cases} u \to \{\alpha_0(u), \alpha_1(u), \dots \alpha_{n-1}(u)\} & \text{if} \quad w \neq 0 \\ u \to \{\alpha_0(u), \alpha_2(u), \dots \alpha_n(u)\} & \text{if} \quad w = 0. \end{cases}$$

In both cases, the linear form α_i satisfies $\alpha_i(u \circ \varphi) = a^{-i}\, \alpha_i(u)$. Besides, we can identify $\mathcal{L}_P$ with $\mathbb{C}$ via the map given by

$$\begin{cases} u \to \alpha_{n-1}(u) & \text{if} \quad w \neq 0 \\ u \to \alpha_n(u) & \text{if} \quad w = 0. \end{cases}$$

Hence the action of $\langle\varphi\rangle$ on $\mathcal{L}_P$ is the character $\chi^{-(n-1)}$ if $w \neq 0$, and χ^{-n} if $w = 0$. To get a trivial action on $\mathcal{L}_P$, we multiply it by χ^{n-1} if $w \neq 0$, and by χ^n if $w = 0$. This yields the first part of (i).

For the second part, let us put $z = \dfrac{w}{n} - \dfrac{(n-1)b}{2}\cdot$ The divisor

$$D = z + (z + b) + \dots + (z + (n-1)b)$$

is $\langle b\rangle$-invariant, and linearly equivalent to $(n-1)\,0 + w$ so that $\mathcal{L}$ is isomorphic to $[D]$. We have an isomorphism between $\mathrm{H}^0(C, \mathcal{L})$ and $\mathbb{C}^n$ given by the list of the residues at the points of D. Hence the representation of $\langle b\rangle$ on $\mathrm{H}^0(C, \mathcal{L})$ is isomorphic to the representation of $\langle b\rangle$ on $\mathbb{C}^n$ that associates to b the matrix of the permutation $(1, 2, \dots, n-1)$. This yields the second part of (i).

If $\Gamma = \{0\}$, the group of solutions G_w is the set of elements (a, b) in the affine group of the form $(1, w/n)(a, 0)(1, -w/n)$ for a in $\mathbb{C}^\times$, it is therefore conjugate of the standard torus $\mathbb{C}^\times$ of scalar multiplications in the affine group. Let $\pi \colon \mathbb{P}^1 \to C$ be the normalization map, it is a set-theoretic bijection. Then $\pi^{-1}\mathcal{L}$ is the subsheaf of the holomorphic line bundle $(n-1)[0] + [w]$ on $\mathbb{P}^1$ consisting of sections s such that $s - s(\infty)$ vanishes at order 2 at ∞. Let μ denote the action of G_w on $\mathbb{P}^1$. We define the action of G_w on $\pi^{-1}\mathcal{L}$ in the natural way as follows: for any $g = (a, b)$ in G_w, the isomorphism $\psi_g \colon \mu(g)^{-1}(\pi^{-1}\mathcal{L}) \to \pi^{-1}\mathcal{L}$ that gives the G_w-structure is given by

$$\psi_g(f)(t) = f(at + b) \times \frac{(at + b)^{n-1}(at + b - w)}{t^{n-1}(t - w)}.$$

Remark that

$$\frac{(t + b/a)^{n-1}(t - (w - b)/a)}{t^{n-1}(t - w)} \underset{t\to\infty}{\sim} 1 + \frac{(nb - w)/a + w}{t} + \mathrm{O}(t^{-2}) = 1 + \mathrm{O}(t^{-2})$$

so the G_w action is well-defined on the sheaf $\pi^{-1}\mathcal{L}$. A basis for $\mathrm{H}^0(\mathbb{P}^1, \pi^{-1}\mathcal{L})$ is given by the sections $s_i = \dfrac{(t - w/n)^i}{t^{n-1}(t - w)}$ for $0 \leq i \leq n$, $i \neq n-1$. Then for $g = (a, b)$ in G_w, we have $g^* s_i = a^i s_i$. □

Corollary 3.9. *Let C be an irreducible reduced curve of arithmetic genus 1, and let $\mathcal{L}$ be a holomorphic line bundle on C such that $n = \deg \mathcal{L} > 0$. Let φ be in $\mathcal{G}_{\mathcal{L}}$, let a be the multiplier of φ, and assume that φ acts on $\mathcal{L}$. Let $\mu(\varphi)$ denote the action of φ on $\mathrm{H}^0(C, \mathcal{L})$.*

(i) Assume that C is smooth.

– *If* $a \neq 1$*, let* P *be a fixed point of* φ*, and let* β *be the action of* φ *on the fiber* $\mathcal{L}_P$*. Then* $\mu(\varphi)$ *is diagonalizable with eigenvalues*

$$\begin{cases} \beta, \beta a, \ldots, \beta a^{n-2}, \beta a^n & \text{if} \quad \mathcal{L} \sim n[p] \\ \beta, \beta a, \ldots, \beta a^{n-1} & \text{otherwise.} \end{cases}$$

– *If* $a = 1$ *and if the translation factor of* φ *is a primitive* ℓ*-torsion point* ($where$ $\ell | n$)*, then* $\mu(\varphi)$ *is diagonalizable, and there exists a complex number* β *such that the eigenvalues of* $\mu(\varphi)$ *are* $\beta, \beta\omega, \ldots, \beta\omega^{\ell-1}$ *where* ω *is any primitive* ℓ*-root of unity.*

(ii) If C *is cuspidal, let* a *be the multiplier of* φ*. If* β *is the action of* φ *on the fiber* $\mathcal{L}_P$ *where* P *is the fixed point of* C^{reg} *under* $\mathcal{G}_{\mathcal{L}}$*, then* $\mu(\varphi)$ *is diagonalizable with eigenvalues* $\beta, \beta a, \beta a^2, \ldots, \beta a^{n-2}, \beta a^n$*.*

Proof. Since C is compact, $\mathrm{Aut}\,(\mathcal{L}) = \mathbb{C}^{\times}$. Therefore two different actions of φ on $\mathcal{L}$ differ by a scalar. The result follows from Proposition 3.8. □

Before stating the main result, we prove a technical lemma:

Lemma 3.10. *Let* C *be a cuspidal curve of arithmetic genus* 1*. Then* $\mathrm{H}^0(C, \Omega^1_C)$ *and* $\mathrm{H}^1(C, \Omega^1_C)$ *are two-dimensional vector spaces. For any automorphism* φ *of* C*, if* a *denotes the multiplier of* φ*, then the eigenvalues of the action of* φ *on* $\mathrm{H}^0(C, \Omega^1_C)$ *and* $\mathrm{H}^1(C, \Omega^1_C)$ *are* $\{a^{-5}, a^{-7}\}$ *and* $\{1, a^{-1}\}$ *respectively.*

Proof. The curve C is isomorphic to the standard cuspidal cubic in the projective plane $\mathbb{P}^2$ given in affine coordinates by the equation $y^2 = x^3$. We introduce the coordinate t such that $x = t^{-2}$ and $y = t^{-3}$. We denote by P_0 be the point at infinity = $[0 : 1 : 0]$ of C, it corresponds to $t = 0$. We also denote by Q the cusp. The sheaf Ω^1_C is locally free of rank one away from the cusp. The maximal torsion subsheaf $\mathcal{T}$ of Ω^1_C is generated by the section $\tau = 2x dy - 3y dx$. The annihilator of τ is the ideal generated by x^2 and y, so $\mathrm{H}^0(C, \mathcal{T})$ is the two-dimensional vector space generated by the sections τ and $x\tau$.

An automorphism φ of $\mathrm{Aut}\,(C)$ given by $t \to at + b$ is induced by the regular map

$$(x, y) \to \left(\frac{a^2 x - 2aby + b^2 x^2}{(a^2 - b^2 x)^2}, \frac{a^3 y - 3a^2 b x^2 + 3ab^2 xy - b^3 y^2}{(a^2 - b^2 x)^3} \right)$$

in the affine coordinates (x, y). By direct computation, the action of the automorhism φ on the two dimensional vector space $(\Omega^1_C)_{|Q}$ in the basis $(dx_{|Q}, dy_{|Q})$ is $a^{-3} \times \begin{pmatrix} a & -2b \\ 0 & 1 \end{pmatrix}$.

For any nonzero complex number ζ, let G_ζ denote the set of φ in $\mathrm{Aut}\,(C)$ fixing $w/2$, that is such that $a\zeta + 2b = \zeta$ where φ is given away from the cusp by the affine map $t \to at + b$. Then G_ζ is isomorphic to $\mathbb{C}^{\times}$, and $\bigcup_{\zeta \in \mathbb{C}^{\times}} G_\zeta$ is dense in the affine group $\mathrm{Aut}\,(C)$. Hence we can assume without loss of generality that the automorphism φ lies in G_ζ for some ζ in $\mathbb{C}^{\times}$.

Let us consider the morphism of sheaves $\Delta\colon \Omega^1_C \to \mathbb{C}_Q$ given by $\alpha dx + \beta dy \to \alpha(Q) + \zeta\beta(Q)$. Then

$$\begin{aligned}\Delta(\varphi^*(\alpha dx + \beta dy)) &= a^{-3} \times \{(a\alpha(Q) - 2b\beta(Q)) + \zeta\beta(Q)\}\\ &= a^{-3}(a\alpha(Q) - \zeta\beta(Q)) = a^{-3}\Delta(\alpha dx + \beta dy).\end{aligned}$$

Hence the morphism Δ becomes G_ζ-equivariant if we endow $\mathbb{C}_Q$ with the G_ζ action given by multiplication with a^{-3}. It is easy to see that the sequence

$$0 \to \mathcal{O}_C\,(-4P_0 + 3/2\zeta) \to \Omega^1_C/\mathcal{T} \to \mathbb{C}_Q \to 0.$$

obtained by multiplication with $\zeta dx - dy$ is exact. This sequence as well as the exact sequence

$$0 \to \mathcal{T} \to \Omega^1_C \to \Omega^1_C/\mathcal{T} \to 0$$

are naturally G_ζ-equivariant: if φ is an automorphism of C in G_ζ given away from the cusp by the affine map $t \to at + b$, then:

– The action of φ on $\mathcal{T}$ is diagonal with eigenvalues a^{-5} and a^{-7} in the basis $(\tau, x\tau)$.

– The action of φ on a section f of $\mathcal{O}_C(-4P_0 + 3/2\zeta)$ is given by the formula

$$t \to f(at+b) \times \frac{at^4(3 - 2\zeta b - 2\zeta at)}{(at+b)^4(3 - 2\zeta t)} = a \times f(at+b) \times \frac{t^4(at + b - 3/2\zeta)}{(at+b)^4(t - 3/2\zeta)}.$$

For the first item, we use a computer-assisted computation. For the second one, it suffices to compute the quotient $\dfrac{\varphi^*(f(t)(wdx - dy))}{wdx - dy}$ on the regular part of C.

Now $\mathrm{H}^0(C, \Omega^1_C)$ is isomorphic as an $\mathrm{Aut}(C)$-module to $\mathrm{H}^0(C, \mathcal{T})$. This gives the first point of the lemma. Besides, $\mathrm{H}^0(C, \Omega^1_C) \simeq \mathrm{H}^0(C, \Omega^1_C/\mathcal{T})$, and we get an $\mathrm{Aut}(C)$-equivariant exact sequence

$$0 \to \mathrm{H}^0(C, \mathbb{C}_Q) \to \mathrm{H}^1(C, \mathcal{O}_C(-4P_0 + 3/2\zeta)) \to \mathrm{H}^1(C, \Omega^1_C) \to 0.$$

Thanks to Serre duality, $\mathrm{H}^1(C, \mathcal{O}_C(-4P_0 + 3/2\zeta)) \simeq \mathrm{H}^0(C, \mathcal{O}_C(4P_0 - 3/2\zeta) \otimes \omega_C)^*$. Note that ω_C is trivial as a holomorphic line bundle, but not as an equivariant holomorphic line bundle; the action of the automorphism group $\mathrm{Aut}(C)$ is given by multiplication with the multiplier. Hence the representation $\mathrm{H}^0(C, \mathcal{O}_C(4P_0 - 3/2\zeta) \otimes \omega_C)$ is given as follows: the action of an element φ of G_ζ on a function f is the function

$$t \to f(at+b) \times \frac{(at+b)^4(t - 3/2\zeta)}{t^4(at + b - 3/2\zeta)}.$$

The action of φ on the equivariant line bundle $\mathcal{O}_C(4P_0 - 3/2\zeta) \otimes \omega_C$ of degree three at the fixed point $\zeta/2$ is the identity, so that thanks to Proposition 3.8 (ii), the eigenvalues of the action of φ on $\mathrm{H}^0(C, \mathcal{O}(4P_0 - 3/2\zeta) \otimes \omega_C)$ are $1, a$ and a^3. Hence the eigenvalues of the action of φ on the contragredient representation $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(4P_0 - 3/2\zeta) \otimes \omega_C)^*$ are $1, a^{-1}$ and a^{-3}. This finishes the proof.

□

Notation: For any unitary polynomial P with nonzero roots, we denote by P^* the unitary polynomial defined by $P^*(t) = t^{\deg(P)} \dfrac{P(t^{-1})}{P(0)}$. If $P(t) = \prod_{i=1}^n (t - \lambda_i)$, then $P^*(t) = \prod_{i=1}^n (t - \lambda_i^{-1})$.

Theorem 3.11. *Let X be a basic rational surface such that $\mathrm{K}_X^2 < 0$ endowed with an automorphism f, and assume that there exists an irreducible curve C in $|-\mathrm{K}_X|$ (such a curve is automatically fixed by f). Let P_f, Q_f and θ_f denote the characteristic polynomials of f^* acting on* $\mathrm{Pic}(X)$, $\mathrm{H}^1(X, \mathrm{T}X)$ *and* $\mathrm{H}^0(C, \mathrm{N}^*_{C/X})$ *respectively.*

(i) If C is cuspidal, $Q_f^*(x) = \dfrac{P_f(x)}{(x-1)(x-a_f^{-1})} \times \theta_f(x) \times (x - a_f^{-5}) \times (x - a_f^{-7})$.

(ii) If C is smooth, let a_f be the action of f^* on the complex line $\mathrm{H}^0(C, \Omega^1_C)$*. Then*

$$Q_f^*(x) = \frac{P_f(x)}{x-1} \times \theta_f(x) \times (x - a_f).$$

Remark 3.12. Since the canonical class is fixed by f, $(x-1)$ always divides P_f. In the cuspidal case, we don't know if $(x - a_f^{-1})$ always divides P_f, but this will be the case in the forthcoming examples of McMullen.

Proof. For classical tools in algebraic geometry concerning curves and curves on surfaces used in this proof (*e.g.* arithmetic and geometric genus, normal and conormal sheaves, dualizing sheaf, Riemann-Roch theorem), we refer the reader to [2, Chap. II].

Let us consider the long exact sequence of cohomology associated with the short exact sequence

$$0 \longrightarrow \Omega^1_X(-C) \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X|C} \longrightarrow 0$$

of sheaves on X. According to Serre duality [28, III Cor. 7.7], for $0 \leq j \leq 2$,

$$\mathrm{H}^j(X, \mathrm{T}X)^* \simeq \mathrm{H}^{2-j}(X, \Omega^1_X \otimes \mathrm{K}_X) \simeq \mathrm{H}^{2-j}(X, \Omega^1_X(-C)).$$

We have $\mathrm{h}^2(X, \mathrm{T}X) = 0$ and

$$\mathrm{h}^2(X, \Omega^1_X) \simeq \mathrm{h}^0(X, \mathrm{T}X \otimes \mathrm{K}_X) = \mathrm{h}^0(X, \mathrm{T}X(-C)) \leq \mathrm{h}^0(X, \mathrm{T}X) = 0.$$

Hence we have an exact sequence of f-equivariant complex vector spaces

$$0 \longrightarrow \mathrm{H}^0(C, \Omega^1_{X|C}) \longrightarrow \mathrm{H}^1(X, \mathrm{T}X)^* \longrightarrow \mathrm{H}^{1,1}(X) \xrightarrow{\pi} \mathrm{H}^1(C, \Omega^1_{X|C}) \longrightarrow 0. \tag{3.1}$$

We can now write down the conormal exact sequence [28, II Prop. 8.12]:

$$0 \longrightarrow \mathrm{N}^*_{C/X} \longrightarrow \Omega^1_{X|C} \longrightarrow \Omega^1_C \longrightarrow 0 \tag{3.2}$$

where the injectivity of the first arrow holds because its kernel is a torsion subsheaf of $\mathrm{N}^*_{C/X}$, the latter being locally free. Since the arithmetic genus of C is 1, the dualizing sheaf ω_C is trivial. Besides, the conormal bundle $\mathrm{N}^*_{C/X}$ has degree $-\mathrm{K}_X^2$ which is positive by assumption, so that combining Riemann-Roch and Serre duality [27, p. 82–83], we get $\mathrm{h}^0(C, \mathrm{N}^*_{C/X}) = -\mathrm{K}_X^2$ and $\mathrm{h}^1(C, \mathrm{N}^*_{C/X})$ vanishes. Therefore $\mathrm{H}^1(C, \Omega^1_{X|C}) \simeq \mathrm{H}^1(C, \Omega^1_C)$. It follows that π can be identified with the pullback morphism from $\mathrm{H}^1(X, \Omega^1_X)$ to $\mathrm{H}^1(C, \Omega^1_C)$ induced by the injection of the curve C in X. We have two f-equivariant exact sequences

$$\begin{cases} 0 \longrightarrow \mathrm{H}^0(C, \Omega^1_{X|C}) \longrightarrow \mathrm{H}^1(X, \mathrm{T}X)^* \longrightarrow \mathrm{H}^{1,1}(X) \longrightarrow \mathrm{H}^1(C, \Omega^1_C) \longrightarrow 0 \\ 0 \longrightarrow \mathrm{H}^0(C, \mathrm{N}^*_{C/X}) \longrightarrow \mathrm{H}^0(C, \Omega^1_{X|C}) \longrightarrow \mathrm{H}^0(C, \Omega^1_C) \longrightarrow 0 \end{cases}$$

If C is smooth, this yields $Q^*_f(x) \times (x-1) = P_f(x) \times \theta_C(x) \times (x - a_f)$. If C is cuspidal, Lemma 3.10 gives

$$Q^*_f(x) \times (x-1) \times (x - a_f^{-1}) = P_f(x) \times \theta_C(x) \times (x - a_f^{-5}) \times (x - a_f^{-7}).$$

□

3.2.2. *Cycle of rational curves.* We now investigate the case of effective anticanonical divisors given by a cycle of smooth rational curves and leave the three other exceptional configurations to the reader.

Remark that if f is an automorphism of a rational surface with positive entropy, then $|-\mathrm{K}_X|$ is either empty or consists of a single divisor (otherwise f would preserve a rational fibration). In this last case, by replacing f by an iterate, we can assume that all irreducible components of the divisor are globally invariant by f.

Theorem 3.13. *Let X be a basic rational surface without nonzero holomorphic vector field. Assume that there exists a reducible and reduced effective anticanonical cycle $D = \sum_{i=1}^r D_i$ on X such that $D_i^2 < 0$ for all i. For any automorphism f of X leaving each D_i globally invariant, let P_f, Q_f and $\theta_{i,f}$ denote the characteristic polynomials of f^* acting on* $\mathrm{Pic}(X)$, $\mathrm{H}^1(X, \mathrm{T}X)$ *and* $\mathrm{H}^0(D_i, \mathrm{N}^*_{D_i/X}(-S_i))$ *respectively, where $S_i = \mathrm{sing}(D) \cap D_i$. Then*

$$Q^*_f(x) = \frac{P_f(x)}{(x-1)^r} \times \prod_{i=1}^r \theta_{i,f}(x).$$

Proof. Let S be the singular locus of D. Then we have an exact sequence

$$0 \longrightarrow \Omega^1_{X|D} \longrightarrow \oplus_{i=1}^r \Omega^1_{X|D_i} \longrightarrow \Omega^1_{X|S} \longrightarrow 0.$$

Since $\mathrm{N}^*_{D_i/X}$ has positive degree, $\mathrm{H}^1(D_i, \Omega^1_{X|D_i}) \simeq \mathrm{H}^1(D_i, \Omega^1_{D_i}) \simeq \mathbb{C}$. Besides, as $\mathrm{H}^0(D_i, \Omega^1_{D_i}) = 0$, we obtain the isomorphism $\mathrm{H}^0(D_i, \Omega^1_{X|D_i}) \simeq \mathrm{H}^0(D_i, \mathrm{N}^*_{D_i/X})$. It follows that

$$\mathrm{H}^0(D, \Omega^1_{X|D}) \simeq \bigoplus_{i=1}^r \mathrm{H}^0(D, \mathrm{N}^*_{D_i/X}(-S_i))$$

and that the map from $\bigoplus_{i=1}^r \mathrm{H}^0(D_i, \Omega^1_{X|D_i})$ to $\mathrm{H}^0(S, \Omega^1_{X|S})$ is onto. Therefore

$$\mathrm{H}^1(D, \Omega^1_{X|D}) \simeq \bigoplus_{i=1}^r \mathrm{H}^1\,(D_i, \Omega^1_{X|D_i}) \simeq \bigoplus_{i=1}^r \mathrm{H}^1(D_i, \Omega^1_{D_i}) \simeq \mathbb{C}^r.$$

We now write down the f-equivariant exact sequence (3.1) used in the proof of Theorem 3.11:

$$0 \longrightarrow \bigoplus_{i=1}^r \mathrm{H}^0(D, \mathrm{N}^*_{D_i/X}(-S_i)) \longrightarrow \mathrm{H}^1(X, \mathrm{T}X)^* \longrightarrow \mathrm{H}^{1,1}(X) \longrightarrow \mathbb{C}^r \longrightarrow \mathrm{H}^0(X, \mathrm{T}X)^* \longrightarrow 0.$$

Since we assumed that there were no nonzero holomorphic vector fields on X, we get

$$Q^*_f(x) \times (x-1)^r = P_f \times \prod_{i=1}^r \theta_{i,f}(x).$$

□

3.3. **Examples.** We have chosen three main examples of application of Theorem 3.11, but this doesn't exhaust the list of possibilities. For instance, other cases of quadratic Cremona transformations leaving a cubic invariant can be dealt with, *e.g.* [14, Ex. 3.3 and Th. 4.4].

3.3.1. *Quadratic transformations fixing a cuspidal cubic.* We start by recalling the construction of automorphisms of rational surfaces of positive entropy obtained from quadratic transformations of the projective plane fixing a cuspidal cubic. This construction goes back to McMullen [37], but we will follow [14].

Let $\mathscr{C}$ be the cuspidal cubic in $\mathbb{P}^2$ given in homogeneous coordinates $(x : y : z)$ by the equation $y^2z = x^3$, and let $P_0 = (0 : 1 : 0)$. The stabilizer in $\mathrm{PGL}(3;\mathbb{C})$ of the cubic $\mathscr{C}$ is isomorphic to $\mathbb{C}^{\times}$, each parameter δ in $\mathbb{C}^{\times}$ corresponding to the linear transformation $(x : y : z) \to (\delta x : y : \delta^3 z)$. There is a natural algebraic group structure on $\mathscr{C}^{\mathrm{reg}}$ with P_0 as origin compatible with the group structure on $\mathrm{Pic}^0(\mathscr{C})$, and the isomorphism $t \to (t : 1 : t^3)$ between $\mathbb{C}$ and $\mathscr{C}^{\mathrm{reg}}$ is an isomorphism of algebraic groups (*see* [28, Ex. 6.11.4]).

Let τ be a permutation of the set $\{1, 2, 3\}$, and let n_1, n_2, n_3 be three integers. The set $\{\tau, n_1, n_2, n_3\}$ is called an *orbit data* (*see* [3]). If $|\tau|$ is the order of τ in the symmetric group $\mathfrak{S}_3$, we define two polynomials p_τ and P_τ as follows:

$$\begin{cases} p_\tau(x) = 1 - 2x + \sum_{j=\tau(j)} x^{1+n_j} + \sum_{j\neq\tau(j)} x^{n_j}(1-x) \\ P_\tau(x) = x^{1+n_1+n_2+n_3}\, p_\tau(x^{-1}) + (-1)^{|\tau|} p_\tau(x) \end{cases}$$

The degree of P_τ is $n_1 + n_2 + n_3 + 1$, and $P_\tau^* = P_\tau$. Let f be a quadratic birational transformation of $\mathbb{P}^2$, and assume that the base locus of f consists of three distinct points p_1^+, p_2^+ and p_3^+. If Δ_{ij} denotes the line between p_i^+ and p_j^+, then the base locus of f^{-1} consists of the three points $p_1^- = f(\Delta_{23})$, $p_2^- = f(\Delta_{13})$ and $p_3^- = f(\Delta_{12})$. We say that f *realizes the orbit data* if for any j with $1 \leq j \leq 3$, the orbit $\{f^k(p_j^-)\}_{0\leq k<n_j-1}$ consists of n_j pairwise distinct points outside the base loci of f and f^{-1} and if $f^{n_j-1}(p_j^-) = p_{\tau(j)}^+$. In this case, the birational map f lifts to an automorphism of the rational surface obtained by blowing up the $n_1 + n_2 + n_3$ points corresponding to the orbits of the points p_j^-. The corresponding characteristic polynomial for the action on the Picard group of this surface is P_τ (*see* [14, §2.1]). We say that an orbit data is *admissible* if it satisfies the following conditions:

$$\begin{cases} \text{If } n_1 = n_2 = n_3 \text{ then } \tau = \mathrm{id}. \\ \text{If } n_i = n_j \text{ for } i \neq j \text{ either } \tau(i) \neq j \text{ or } \tau(j) \neq i. \\ \text{All } n_i \text{ are at least } 3 \text{ and one is at least } 4. \end{cases}$$

Proposition 3.14 ([14, Thm. 1 & 3])**.**

(i) Let p_1^+, p_2^+ and p_3^+ be three points in $\mathscr{C}^{\mathrm{reg}}$ such that $p_1^+ + p_2^+ + p_3^+ \neq 0$, and let μ be a given number in $\mathbb{C}^{\times}$. Then there exists a unique quadratic transformation f of $\mathbb{P}^2$ fixing $\mathscr{C}$, having p_1^+, p_2^+ and p_3^+ as base points, and such that $f_{|\mathscr{C}}$ has multiplier μ. Besides, the translation factor of $f_{|\mathscr{C}}$ is $\epsilon = \frac{1}{3}(p_1^+ + p_2^+ + p_3^+)\,\mu$ and the points p_j^- are given by $p_j^- = \mu p_j^+ - 2\epsilon$ for $1 \leq j \leq 3$.

(ii) Let (τ, n_1, n_2, n_3) be an admissible orbit data, and assume furthermore that μ is a root of P that is not a root of unity. Then there exist p_1^+, p_2^+ and p_3^+ with $p_1^+ + p_2^+ + p_3^+ \neq 0$ such

that the quadratic map f given in (i) realizes the orbit data (τ, n_1, n_2, n_3) and has multiplier μ. Besides, such a quadratic map is unique modulo conjugation by the centralizer of $\mathscr{C}$ in $\mathrm{PGL}(3;\mathbb{C})$.

Proposition 3.15. *Let p_1^+, p_2^+ and p_3^+ be three points in $\mathscr{C}^{\mathrm{reg}}$ such that $p_1^+ + p_2^+ + p_3^+ \neq 0$, let (τ, n_1, n_2, n_3) be an admissible orbit data, let μ be a root of P_τ that is not a root of unity, and let f be the corresponding birational quadratic map given by Proposition 3.14. Then the eigenvalues of* $\mathrm{d}f$ *at the unique fixed point on $\mathscr{C}^{reg}$ are μ and $\mu^{3-n_1-n_2-n_3}$.*

Proof. Due to a lack of conceptual arguments, we provide a direct proof by calculation, which is computer assisted. We sketch the argument and refer to the Maple file for computational details.

Since the stabilizer of $\mathscr{C}$ in $\mathrm{PGL}(3;\mathbb{C})$ acts transitively on $\mathscr{C}^{\mathrm{reg}}$, we can assume without loss of generality that $p_1^+ = (1 : 1 : 1)$. We put $p_2^+ = (\alpha : 1 : \alpha^3)$ and $p_3^+ = (\beta : 1 : \beta^3)$ and assume that $\alpha + \beta \neq -1$. If $\sigma \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the Cremona involution given by $\sigma(x : y : z) = (yz : xz : xy)$ and N is the linear transformation sending the three points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ to p_1^+, p_2^+ and p_3^+ respectively, then the base locus of the quadratic transformation $\sigma \circ N$ consists of the three points $\{p_i^+\}_{1\leq i\leq 3}$. It is then possible to find a matrix M such that $f = M \circ \sigma \circ N$ fixes the cuspidal cubic $\mathscr{C}$. Up to multiplying by an element of the centralizer of $\mathscr{C}$ in $\mathrm{PGL}(3;\mathbb{C})$, we can arrange that the multiplier of $f_{|\mathscr{C}}$ is μ.

The translation factor is $\epsilon = \frac{1}{3}(\alpha+\beta+1)\,\mu$, so that the unique point on $\mathscr{C}^{\mathrm{reg}}$ fixed by f is $p = \frac{(\alpha+\beta+1)\mu}{3(1-\mu)}$. We have $p_1^- = \frac{1}{3}(1 - 2\alpha - 2\beta)\,\mu$, $p_2^- = \frac{1}{3}(-2 + \alpha - 2\beta)\,\mu$ and $p_3^- = \frac{1}{3}(-2 - 2\alpha + \beta)\,\mu$. We now write down the conditions in order that f realizes the orbit data. We have $f^n(t) = \mu^n(t - p) + p$, so that we must solve the equations

$$\mu^{n_i-1}(p_i^- - p) + p = p_i^+ \quad \text{for} 1 \leq i \leq 3.$$

If we take two of these three equations, we have two affine equations in the variables α and β that give uniquely α and β as rational fractions in μ. The third equation is automatically satisfied if μ is a root of P_τ. Then we can compute the eigenvalues of $\mathrm{d}f$ at p: the multiplier μ is of course an eigenvalue, and the other one ζ is a rational fraction in μ. Now an explicit calculation yields the formula $\zeta\,\mu^{n_1+n_2+n_3-3} = 1$. □

Theorem 3.16. *Let (τ, n_1, n_2, n_3) be an admissible orbit data, let μ be a root of P_τ that is not a root of unity, and let f be a birational quadratic map realizing the orbit data (τ, n_1, n_2, n_3), fixing the cuspidal cubic $\mathscr{C}$, and having multiplier μ when restricted to the cubic $\mathscr{C}$. If X is the corresponding rational surface and if g is the lift of f as an automorphism of X, then* $\mathrm{m}(X, g) \leq 3-|\tau|$. *In particular, if τ has order three, then g is rigid.*

Proof. Recall that X is obtained by blowing up $n_1 + n_2 + n_3$ points, and that the conormal bundle of the strict transform of $\mathscr{C}$ in X has degree $n_1 + n_2 + n_3 - 9$. Thanks to Theorem 3.11 (a), Corollary 3.9 (ii) and Proposition 3.15, the characteristic polynomial Q_g of g acting on $\mathrm{H}^1(X, \mathrm{T}X)$ is given by

$$Q_g^*(x) = \frac{P_\tau(x)}{(x-1)(x-\mu^{-1})} \times \left(\prod_{\substack{j=6 \\ j\neq 7}}^{n_1+n_2+n_3-3} (x - \mu^{-j}) \right) \times (x - \mu^{-5}) \times (x - \mu^{-7}).$$

and using the fact that $P_\tau = P_\tau^*$, we get

$$Q_g(x) = \frac{P_\tau(x)}{(x-1)(x-\mu)} \times \prod_{j=5}^{n_1+n_2+n_3-3} (x-\mu^j).$$

We look at the multiplicity of the root 1 in Q_g. Since μ is not a root of unity, we can reduce the problem to the polynomial P_τ. We claim that 1 occurs with multiplicity $4 - |\tau|$. We deal with the three different cases.

- If $\tau = (1)(2)(3)$, $P_\tau(1) = P'_\tau(1) = P^{(2)}_\tau(1) = 0$, and $P^{(3)}_\tau(1) = 6n_1n_2n_3\left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - 1\right) < 0$.
- If $\tau = (12)(3)$, $P_\tau(1) = P'_\tau(1) = 0$, and $P^{(2)}_\tau(1) = 2[4n_3 - (n_1+n_2)(n_3-1)] \neq 0$. Now $n_3 - 1$ cannot divide $4n_3$ except if $n_3 = 3$ or $n_3 = 5$. If $n_3 = 3$, we would have $n_1 + n_2 = 6$, which is not possible. If $n_3 = 5$, we get $n_1 + n_2 = 5$ which is also excluded.
- If $\tau = (123)$, $P_\tau(1) = 0$, and $P'_\tau(1) = 9 - n_1 - n_2 - n_3 < 0$.

□

3.3.2. *The construction of Blanc and Gizatullin.* We start by a brief recollection of the construction of automorphisms fixing a smooth elliptic curve in [5].

Let $\mathscr{C}$ be a smooth cubic in $\mathbb{P}^2$. For any point p in $\mathscr{C}$, let σ_p be the birational involution of $\mathbb{P}^2$ defined as follows: if ℓ is a generic line of $\mathbb{P}^2$ passing through p, $(\sigma_p)_{|\ell}$ is the involution of ℓ fixing the two other intersection points of ℓ with $\mathscr{C}$. The involution σ_p has five distinct[7] base points (including p), and becomes an automorphism on the rational surface obtained by blowing up these five points. We denote the set of five base points of σ_p by S_p. If $S_p = \{p, p_1, p_2, p_3, p_4\}$, then the p_i's are the four points of $\mathscr{C}$ such that (pp_i) is tangent to $\mathscr{C}$ at p_i. Let $E_p, E_{p_1}, E_{p_2}, E_{p_3}, E_{p_4}$ be the exceptional divisors of the blowup. The lift of σ_p maps E_p to the unique conic of $\mathbb{P}^2$ passing through all points of S_p and each E_{p_i} to the line (pp_i). Thus the action of σ_p on the Picard group of the blown-up surface is given by

$$\begin{cases} E_p \to 2H - E_p - E_{p_1} - E_{p_2} - E_{p_3} - E_{p_4} \\ E_{p_i} \to H - E_p - E_{p_i} \quad 1 \leq i \leq 4 \\ H \to 3H - 2E_p - E_{p_1} - E_{p_2} - E_{p_3} - E_{p_4} \end{cases} \tag{3.3}$$

where H is the pull-back of the hyperplane class (*see* [5, Lemma 17]).

Let us now fix three points p, q, r on $\mathscr{C}$ such that the set S_p, S_q and S_r do not overlap (the construction works with arbitrary many points and without the genericity condition, *see* [5] for further details). Then σ_p, σ_q and σ_r lift to automorphisms of the rational surface X obtained by blowing up $\mathbb{P}^2$ in the 15 points S_p, S_q, S_r (we will still denote the lifts by σ_p, σ_q and σ_r) and provides an embedding of the free product $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$ in $\mathrm{Aut}\,(X)$.

This construction can be made in families: the family of smooth cubics in $\mathbb{P}^2$ with three (ordered) distinct marked points is a smooth quasi-projective variety $\mathcal{V}$ of dimension $9 + 3 = 12$. We have a natural deformation $\mathfrak{X}$ of rational surfaces over $\mathcal{V}$: for any point (C, p, q, r) in $\mathcal{V}$, the corresponding

7. If we choose p as the origin of $\mathscr{C}$, then the four other base points are the 2-torsion points of $\mathscr{C}$.

rational surface X is the blowup of the projective plane $\mathbb{P}^2$ at S_p, S_q and S_r. Besides, $\mathfrak{X}$ is endowed with three involutions $\mathfrak{s}_p$, $\mathfrak{s}_q$ and $\mathfrak{s}_r$.

Let $\Psi = \mathfrak{s}_p \circ \mathfrak{s}_q \circ \mathfrak{s}_r$; it is an automorphism of $\mathfrak{X}$. For any v in $\mathcal{V}$, the characteristic polynomial of Ψ_v^* acting on the Picard group of $\mathfrak{X}_v$ is $P_{\Psi_v} = (t^2 - 18t + 1)(t-1)^4(t+1)^{10}$ so that the first dynamical degree of Ψ_v is $9 + 4\sqrt{5}$. In particular Ψ_v has positive entropy.

Theorem 3.17. *The family* $(\mathfrak{X}, \Psi)$ *is complete at all of its fibers.*

Proof. The three involutions σ_p, σ_q and σ_r fix pointwise the strict transform C of the curve $\mathscr{C}$, and act by multiplication by -1 on any fiber of the conormal bundle $\mathrm{N}^*_{C/X}$. Thus for any point v in the base $\mathcal{V}$, $\Theta_{\Psi_v} = (t+1)^6$ and $a_{\Psi_v} = 1$. Hence we get by Theorem 3.11 the formula

$$Q_{\Psi_v} = (t^2 - 18t + 1)(t-1)^4(t+1)^{16}$$

so that $\dim \ker(\Psi_v^* - \mathrm{id}) = 4$. The base $\mathcal{V}$ of $\mathfrak{X}$ is a submanifold of $S^{\dagger}_{15}$ of dimension 12 which is $\mathrm{PGL}(3;\mathbb{C})$-invariant. By Proposition 3.1, the Kodaira-Spencer map of $(\mathfrak{X}, \Psi)$ has rank 4 at every point. It is therefore surjective, and the result follows from Theorem 2.7. □

3.3.3. *Unnodal Halphen surfaces.* We start with a short reminder about Halphen surfaces. We refer to [11, Prop. 2.1] and [23, §7] for more details.

Definition 3.18. A Halphen surface of index m is a rational surface X such that $|-m\mathrm{K}_X|$ has no fixed part and defines a base point free pencil.

If X is a Halphen surface of index m, $\mathrm{K}_X^2 = 0$ and the generic fiber of the pencil $-m\mathrm{K}_X$ is a smooth elliptic curve. In fact, Halphen surfaces are exactly the minimal rational surfaces. They are obtained by blowing up a pencil of curves of degree $3m$ in $\mathbb{P}^2$ with 9 base points [8] along the base locus of the pencil, and $|-m\mathrm{K}_X|$ is the strict transform of the pencil. If $m = 1$, all members of the pencil are anticanonical divisors. If $m \geq 2$, the Riemann-Roch theorem yields the existence of an anticanonical divisor $\mathfrak{D}$, and the only multiple fiber of the elliptic pencil is $m\mathfrak{D}$. Besides, $\mathrm{N}^*_{\mathfrak{D}/X}$ is a torsion point of index m in $\mathrm{Pic}(\mathfrak{D})$.

Let us explain a concrete way (given at the end of [11, §2.2], *see also* [31, §10.5.1]) to construct Halphen surfaces directly without Halphen pencils. For simplicity, we will only consider the case where the anticanonical divisor is smooth and reduced. Let $\mathscr{C}$ be a smooth cubic in $\mathbb{P}^2$, and consider nine pairwise distinct points [9] $p_1, \ldots, p_9$ on $\mathscr{C}$. Let $\mathfrak{h}$ be the class $\mathcal{O}_{\mathbb{P}^2}(1)_{|\mathscr{C}}$ in $\mathrm{Pic}(\mathscr{C})$. We define X as the blowup of $\mathbb{P}^2$ at the nine points p_i. Let $\mathfrak{d} = 3\mathfrak{h} - \sum_{i=1}^{9} [p_i]$, and let C be the strict transform of $\mathscr{C}$ in X. Considering $\mathfrak{d}$ as a divisor on C, $\mathfrak{d} = \mathrm{N}^*_{C/X}$ in $\mathrm{Pic}^0(C)$. If $m\mathfrak{d} = 0$, then X is a Halphen surface of index m. Let o be an inflexion point of $\mathscr{C}$, so that $\mathfrak{h} \sim 3o$. Choosing o as an origin in $\mathscr{C}$ and denoting by $\oplus$ the group law in $\mathscr{C}$, the condition $m\mathfrak{d} = 0$ in $\mathrm{Pic}^0(C)$ means that $m(p_1 \oplus \ldots \oplus p_9) = o$ in $\mathscr{C}$.

We can put this construction in families. We must distinguish the cases $m = 1$ and $m \geq 2$, which behave slightly differently.

8. Such a pencil is called a Halphen pencil of index m. If $m = 1$, this is an ordinary pencil of cubics in the plane.
9. The construction works also for infinitely near points, but all points p_i must be based on the various strict transforms of $\mathscr{C}$.

If $m \geq 2$, let $\mathcal{U}_m$ be the set of pairs $\{\mathscr{C}, p_1, \ldots, p_9\}$ where $\mathscr{C}$ is a smooth cubic, the p_i's lie on $\mathscr{C}$ and are pairwise distinct, and $m\mathfrak{d} = 0$ in $\mathrm{Pic}^0(\mathscr{C})$; this is a smooth quasi-projective variety of dimension $9+9-1 = 17$ with a natural action of $\mathrm{PGL}(3;\mathbb{C})$. Besides, we have a universal family $\mathfrak{H}_m$ of Halphen surfaces of index m over $\mathcal{U}_m$. Since Halphen surfaces of index $m \geq 2$ have a unique anticanonical divisor, the cubic $\mathscr{C}$ is entirely determined by the points p_i, so that we can see $\mathcal{U}_m$ as a locally closed smooth algebraic variety in the configuration space $(\mathbb{P}^2)^9/\mathfrak{S}_9$ stable by the action of $\mathrm{PGL}(3;\mathbb{C})$.

If $m = 1$, the nine points p_i don't determine the cubic curve $\mathscr{C}$. Let $\mathcal{U}_1$ denote the set of pencils of cubics on $\mathbb{P}^2$ with smooth generic fiber and no infinitely near points in the base locus, which is a Zariski open subset of $\mathbb{P}^{16}$ that can be seen as a smooth locally closed subset of $(\mathbb{P}^2)^9/\mathfrak{S}_9$ via the map associating to each pencil its base locus. The variety $\mathcal{U}_1$ carries a natural family $\mathfrak{H}_1$ of Halphen surfaces of index 1 obtained by blowing up the base locus of the pencils. Besides, $\mathcal{U}_1$ is again stable by the action of $\mathrm{PGL}(3;\mathbb{C})$.

Lastly, let us recall some results about *unnodal* Halphen surfaces. We refer the reader to [11, §2.3] for more details. An Halphen surface is called unnodal if all members of the pencil $|-m\mathrm{K}_X|$ are irreducible. This condition is generic among Halphen surfaces of index m. In particular, every genericity condition we have assumed on the Halphen set is satisfied for unnodal Halphen surfaces. Besides, their automorphisms group admits a particularly nice description: if X is unnodal, the lattice $\mathfrak{L}(X) = (\mathrm{K}_X^{\perp} \cap \mathrm{H}^2(X,\mathbb{Z}))/\mathbb{Z}\mathrm{K}_X$ embeds naturally as a finite-index subgroup of $\mathrm{Aut}(X)$; and this group acts by translation on the fibers of the elliptic fibration. The action of any element α of $\mathfrak{L}(X)$ on the Picard group of X is given by the explicit formula [10]

$$f_\alpha^*(D) = D - m\,(D.\mathrm{K}_X)\,\alpha + \left\{ m\,(D.\alpha) - \frac{m^2}{2}(D.\mathrm{K}_X)\,\alpha^2 \right\} \mathrm{K}_X. \tag{3.4}$$

In particular, f_α^* acts unipotently on $\mathrm{Pic}(X)$, but f_α^* is not of finite order (it is a true parabolic element). Thus X has no nonzero holomorphic vector field.

Proposition 3.19. *Let X be an unnodal Halphen surface of some index m carrying a smooth anticanonical curve, and let α be an element in the lattice $\mathfrak{L}(X)$. If U is a small neighborhood of a point defining X in $\mathcal{U}_m$, let $\mathfrak{f}_\alpha$ be a lift of f_α [11] on the family $(\mathfrak{H}_m)_{|U}$. Then $\{(\mathfrak{H}_m)_{|U}, \mathfrak{f}_\alpha\}$ is a complete deformation of (X, f_α).*

Proof. Let $\{X_t, f_t\}_t$ be a local deformation of the pair (X, f_α). Then for any t, the automorphism f_t remains a parabolic isometry of $\mathrm{H}^2(X_t,\mathbb{Z})$. Therefore, thanks to the main result of [20] (*see* [23]), X_t is a Halphen surface. Now thanks to Proposition 3.1, we can write X_t as the blowup of nine points $p_i(t)$, $1 \leq i \leq 9$ varying holomorphically with t. For t small enough, X_t is anticanonical so the points $p_i(t)$ lie on a plane cubic curve $\mathscr{C}_t$. Since X_t is Halphen, the point $\mathfrak{d}_t$ is a torsion point in $\mathrm{Pic}^0(\mathscr{C}_t)$. We can see the map $t \to \mathfrak{d}_t$ as a local holomorphic section of the Jacobian variety of X_t whose values are torsion elements. Since the order of $\mathfrak{d}_0$ is m, it follows that all $\mathfrak{d}_t$ have order m. Thus the points $p_i(t)$ define an element in $\mathcal{U}_m$, which proves that $\{X_t\}_t$ is obtained by pullback from $\mathfrak{H}_m$ and then $\{X_t, f_t\}_t$ is obtained by pullback from $\{(\mathfrak{H}_m)_{|U}, \mathfrak{f}_\alpha\}$. □

10. There is a sign mistake in [20], also pointed out in [10].

11. The lift $\mathfrak{f}_\alpha$ exists because the lattices $\mathfrak{L}(X)$ form a local system of abelian groups over $\mathcal{U}_m$.

The proof of Proposition 3.19 relies heavily on Gizatullin's result. Let us explain how it is possible to obtain this result (at least for $m \geq 2$) using our method.

Proposition 3.20. *Let X be an unnodal Halphen surface of index m carrying a smooth anticanonical curve. Then there is a natural $\mathfrak{L}(X)$-equivariant exact sequence*

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{H}^1(X, \mathrm{TX})^* \longrightarrow \mathrm{K}_X^{\perp} \longrightarrow 0$$

where $\mathrm{K}_X^{\perp}$ *denotes the orthogonal of the canonical class in* $\mathrm{Pic}\,(X)$*, and* $\mathfrak{L}(X)$ *acts trivially on* $\mathbb{C}$*. In particular the action of* $\mathfrak{L}(X)$ *on* $\mathrm{H}^1(X, \mathrm{T}X)$ *is unipotent and for any* α *in* $\mathfrak{L}(X)$*,*

$$8 \leq \dim\,(f_\alpha^* - \mathrm{id}) \leq 9$$

where f_α^* *is the action of* f_α *on* $\mathrm{H}^1(X, \mathrm{T}X)$*.*

Remark 3.21. Proposition 3.19 is stronger because it gives $\dim\ker\,(f_\alpha^* - \mathrm{id}) = \begin{cases} 8 \text{ if } m = 1 \\ 9 \text{ if } m \geq 2. \end{cases}$

Proof. We adapt the proof of Theorem 3.11 in this situation. To do so, we study the conormal exact sequence (3.1) of C.

Assume that m $\geq$ **2**. Since $\mathrm{N}^*_{C/X}$ is a torsion point of order m in $\mathrm{Pic}^0(C)$, $\mathrm{H}^0(C, \mathrm{N}^*_{C/X}) = \{0\}$, so that by Riemann-Roch $\mathrm{H}^1(C, \mathrm{N}^*_{C/X}) = \{0\}$. Hence we get isomorphisms

$$\mathrm{H}^i(C, \Omega^1_{X|C}) \simeq \mathrm{H}^i(C, \Omega^1_C) \qquad i \in \{0, 1\},$$

so that the kernel of the restriction map $\mathrm{H}^1(X, \Omega^1_X) \to \mathrm{H}^1(C, \Omega^1_{X|C})$ identifies with $\mathrm{K}_X^{\perp}$. Since $\mathfrak{L}(X)$ acts trivially on $\mathrm{H}^0(C, \Omega^1_C)$, the result follows using (3.1).

Assume that m $=$ **1**. For any smooth fiber C_t of the pencil, $\mathrm{N}_{C_t/X}$ is canonically isomorphic to $\mathcal{O}_{C_t} \otimes_{\mathbb{C}} \mathrm{T}_t\mathbb{P}^1$ and the extension class of the normal exact sequence (which is dual to (3.1)) identifies with the Kodaira-Spencer map of the family $\{C_s\}_{|s-t|<\epsilon}$ via the isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_{C_t}}(C_t, \mathcal{O}_{C_t} \otimes_{\mathbb{C}} \mathrm{T}_t\mathbb{P}^1, \mathrm{T}C_t) \simeq \mathrm{H}^1(C_t, \mathrm{T}C_t) \otimes_{\mathbb{C}} \mathrm{T}_t^*\mathbb{P}^1.$$

There are Halphen surfaces of index 1 such that the complex structure of the smooth fibers of the elliptic pencil remains constant (they are described in [20, Prop. B]) but they are not unnodal. Thus, if C is a generic fiber of the elliptic fibration, the Kodaira-Spencer map $\kappa\colon \mathrm{T}\mathbb{P}^1 \to \mathrm{H}^1(C, \mathrm{T}C)$ is nonzero. It follows that (3.1) is not holomorphically split over $\mathcal{O}_C$, so that the maps

$$\mathrm{H}^0(C, \mathrm{N}^*_{C/X}) \to \mathrm{H}^0(C, \Omega^1_{X|C}) \qquad \text{and} \qquad \mathrm{H}^1(C, \Omega^1_{X|C}) \to \mathrm{H}^1(C, \Omega^1_C)$$

are isomorphisms. We conclude using (3.1) again. □

As a corollary, we get a new proof of Proposition 3.19 for $m \geq 2$: indeed, the family $\{(\mathfrak{H}_m)_{|U}, \mathfrak{f}_\alpha\}$ is parameterized by a smooth base of dimension 17 which is stable under the action of $\mathrm{PGL}(3; \mathbb{C})$. Thanks to Proposition 2.2, its Kodaira-Spencer map has rank 9 at any point of U. Then the conclusion follows from Theorem 2.7.

4. Kummer surfaces

By definition, a Kummer surface X is a desingularization of a quotient $\mathcal{A}/G$ where $\mathcal{A}$ is an abelian surface and G is a finite group of automorphisms of A. These subgroups have been classified (*see* [18]), and the geometry of the corresponding Kummer surfaces have been studied in [50]. Many situations can occur. The most famous case is $G = \{\pm \mathrm{id}\}$, and in this case X is a K3 surface. In this part, we will deal with two special pairs $(\mathcal{A}, G)$ such that X is a rational surface. There are many other cases apart these two ones where this happens (*see* [50, Thm 2.1]).

4.1. **Rational Kummer surface associated with the hexagonal lattice.**

4.1.1. *Basic properties.* Let $\mathcal{E}$ be the elliptic curve obtained by taking the quotient of the complex line $\mathbb{C}$ by the lattice $\Lambda = \mathbb{Z}[\mathbf{j}]$ of Eisenstein integers, and let $\mathcal{A}$ be the abelian surface $\mathcal{E} \times \mathcal{E}$, and let ϕ be the automorphism of order 3 defined by $\phi(x, y) = (\mathbf{j}x, \mathbf{j}y)$. Since $\phi^2 = \phi^{-1}$, the automorphisms ϕ and ϕ^2 have the same 9 fixed points which are

$$\begin{cases} p_1 = (0,0) & p_2 = \left(0, \dfrac{2}{3} + \dfrac{\mathbf{j}}{3}\right) & p_3 = \left(0, \dfrac{1}{3} + \dfrac{2\mathbf{j}}{3}\right) \\ p_4 = \left(\dfrac{2}{3} + \dfrac{\mathbf{j}}{3}, 0\right) & p_5 = \left(\dfrac{2}{3} + \dfrac{\mathbf{j}}{3}, \dfrac{2}{3} + \dfrac{\mathbf{j}}{3}\right) & p_6 = \left(\dfrac{2}{3} + \dfrac{\mathbf{j}}{3}, \dfrac{1}{3} + \dfrac{2\mathbf{j}}{3}\right) \\ p_7 = \left(\dfrac{1}{3} + \dfrac{2\mathbf{j}}{3}, 0\right) & p_8 = \left(\dfrac{1}{3} + \dfrac{2\mathbf{j}}{3}, \dfrac{2}{3} + \dfrac{\mathbf{j}}{3}\right) & p_9 = \left(\dfrac{1}{3} + \dfrac{2\mathbf{j}}{3}, \dfrac{1}{3} + \dfrac{2\mathbf{j}}{3}\right) \end{cases}$$

We denote this set by S, it is a subgroup of the 3-torsion points in $\mathcal{A}$. Let G be the group of order 3 generated by ϕ in $\mathrm{Aut}(\mathcal{A})$. Then $\mathcal{A}/G$ is a singular surface, and the nine singularities corresponding to the points of S are of type A_3. Their blowup produces a smooth projective surface X called a rational Kummer surface, and the nine exceptional divisors are of self-intersection -3.

To avoid using singular surfaces, we use a slightly different construction yielding the same surface X: first we blow up the set S in $\mathcal{A}$ and denote by $\widetilde{\mathcal{A}}$ the resulting surface and by $\widetilde{E_i}$ the exceptional divisors corresponding to the points p_i. The group G acts on $\widetilde{\mathcal{A}}$, and the quotient $\widetilde{\mathcal{A}}/G$ is X. For $1 \leq i \leq 9$, let E_i be the image of $\widetilde{E_i}$ in X, it is a rational curve of self-intersection -3. We have the following diagram, where $\delta \colon \widetilde{\mathcal{A}} \to \mathcal{A}$ is the blowup map and $\pi \colon \widetilde{\mathcal{A}} \to X$ is the projection

$$\begin{array}{ccc} & \widetilde{\mathcal{A}} & \\ {}^{\delta}\swarrow & & \searrow^{\pi} \\ A & & X \end{array}$$

We can describe more precisely the map π: $(\widetilde{\mathcal{A}}, \pi)$ is the cyclic covering of X of order 3 branched along the rational curves E_i. In particular, for $1 \leq i \leq 9$, we have $\pi^* E_i = 3\widetilde{E_i}$. Let us recall the following well-known fact (*see* [9, Ex.4 p. 103]):

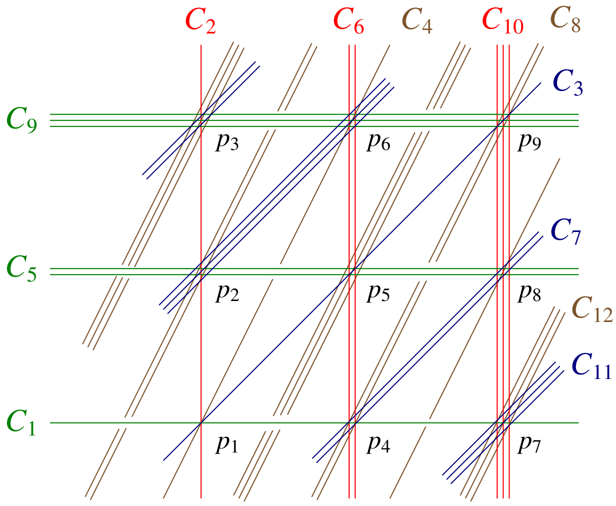
Lemma 4.1. *The surface X is a basic rational surface that can be obtained by blowing* 12 *distinct points in* $\mathbb{P}^2$.

Proof. Let us consider the four following curves $(C_i)_{1\leq i\leq 4}$ in $\mathcal{A}$

$$C_1 = E \times \{0\}, \quad C_2 = \{0\} \times E, \quad C_3 = \Delta_{\mathcal{A}}, \quad C_4 = \Gamma_{-\phi} \tag{4.1}$$

where $\Delta_{\mathcal{A}}$ is the diagonal of $\mathcal{A}$ and $\Gamma_{-\phi}$ is the graph of $-\phi$. Define 8 other curves $(C_i)_{5\leq i\leq 12}$ as follows:

$$\begin{cases} C_5 = C_1 + p_2 & C_6 = C_2 + p_4 & C_7 = C_3 + p_4 & C_8 = C_4 + p_4 \\ C_9 = C_1 + p_3 & C_{10} = C_2 + p_7 & C_{11} = C_3 + p_7 & C_{12} = C_4 + p_7 \end{cases}$$



The strict transforms of the 12 curves C_i $(1 \leq i \leq 12)$ in $\widetilde{\mathcal{A}}$ are ϕ-invariant elliptic curves of self-intersection -3, since each of them pass through exactly three points of S with multiplicity one). Their images by π give 12 smooth rational curves $(\mathscr{E}_i)_{1\leq i\leq 12}$ of self-intersection -1. Blowing down these 12 curves, we get a smooth surface Y. Since $\mathrm{H}^1(\mathcal{A},\mathbb{Z})^G \simeq (\Lambda^* \times \Lambda^*)^G = \{0\}$ [12] where Λ^* denotes the dual lattice of Λ, $\mathrm{b}_1(Y)$ vanishes. Now we compute the Euler characteristic of Y:

$$\chi(Y) = \chi(X) - 12 = \chi(X \smallsetminus \{E_i\}_i) + 6 = \frac{\chi(\widetilde{A} \smallsetminus \{\widetilde{E}_i\}_i)}{3} + 6 = \frac{\chi(\widetilde{A})}{3} = \chi(A) + 3 = 3 = \chi(\mathbb{P}^2)$$

so that $b_2(Y) = 1$. To conclude that the surface Y is isomorphic to $\mathbb{P}^2$ and not to a fake projective plane, it suffices to prove that Y is not a surface of general type (*see* [22, p. 487]). Denoting by κ the Kodaira dimension, we have $\kappa(Y) = \kappa(X) \leq \kappa(\widetilde{\mathcal{A}}) = \kappa(\mathcal{A}) = 0$. This finishes the proof. □

Remark 4.2. After blowing down the 12 exceptional curves, the exceptional divisors E_i are lines in $\mathbb{P}^2$, since they are of self-intersection one. Each point belongs to three lines and each line passes through four points.

The Picard group of X can be described explicitly in the following way: since X is rational, $\mathrm{Pic}(X)$ is isomorphic to $\mathrm{H}^2(X, \mathbb{Z}_X)$. First we compute:

$$\mathrm{H}^2(\widetilde{\mathcal{A}}, \mathbb{Z}_{\widetilde{\mathcal{A}}}) \simeq \mathrm{H}^2(\mathcal{A}, \mathbb{Z}_{\mathcal{A}}) \oplus \left(\bigoplus_{i=1}^{9} \mathbb{Z}[\widetilde{E}_i] \right) \simeq \wedge^2_{\mathbb{Z}}(\Lambda^* \times \Lambda^*) \oplus \left(\bigoplus_{i=1}^{9} \mathbb{Z}[\widetilde{E}_i] \right).$$

12. For any G-module M, we put $M^G = \{m \in M \text{ s. t. } \forall g \in G, g.m = m\}$.

Hence we get an isomorphism of $\mathrm{GL}(2;\Lambda)$-modules:

$$\mathrm{Pic}(X) \simeq \mathrm{H}^2(\widetilde{\mathcal{A}}, \mathbb{Z}_{\widetilde{\mathcal{A}}})^G \simeq \left(\wedge^2_{\mathbb{Z}}(\Lambda^* \times \Lambda^*)\right)^G \oplus \bigoplus_{i=1}^{9} \mathbb{Z}[\widetilde{E}_i]. \tag{4.2}$$

It is easy to see that $\left(\wedge^2_{\mathbb{Z}}(\Lambda^* \times \Lambda^*)\right)^G$ is a free $\mathbb{Z}$-module of rank 4, a basis (over $\mathbb{Q}$) of this module being given by the curves $(C_i)_{1\le i\le 4}$. Thus $\mathrm{Pic}(X)$ is a free $\mathbb{Z}$-module of rank 13 (which is 1 plus the the number of points blown up, as expected). We end this section by a description of the canonical class of X.

Lemma 4.3. *We have* $|-\mathrm{K}_X| = |-2\mathrm{K}_X| = \varnothing$ *and* $|-3\mathrm{K}_X| = \sum_{i=1}^9 E_i$.

Proof. Since π is a cyclic covering, the generalized Riemann-Hurwitz formula for branched cyclic covers gives $\pi^*\mathrm{K}_X = \mathrm{K}_{\widetilde{\mathcal{A}}}(-2\sum_{i=1}^9 \widetilde{E}_i)$. On the other hand, $\mathrm{K}_{\widetilde{\mathcal{A}}} = \delta^*\mathrm{K}_{\mathcal{A}} + \sum_{i=1}^9 \widetilde{E}_i$ so that $\pi^*\mathrm{K}_X = -\sum_{i=1}^9 \widetilde{E}_i$. It follows that $-3\mathrm{K}_X \sim \sum_{i=1}^9 E_i$. Now if D belongs to $|-3\mathrm{K}_X|$, we have $D.E_i = -3$, so that D must contain E_i as a component. Thus $D = \sum_{i=1}^9 E_i + D'$ where D' is effective. But $D' \sim 0$ so that $D' = 0$ and $|-3\mathrm{K}_X| = \sum_{i=1}^9 E_i$. □

4.1.2. *Linear automorphisms of the Kummer surface.* The group $\mathrm{GL}(2;\Lambda)$ acts linearly on $\mathbb{C}^2$ and preserves the lattice $\Lambda \times \Lambda$. Therefore any element M of $\mathrm{GL}(2;\Lambda)$ induces an automorphism f_M on $\mathcal{A}$ that commutes with the automorphism ϕ of A, hence leaves the set S globally invariant. Each f_M lifts to an automorphism $\tilde{f}_M$ of the blown up abelian surface $\widetilde{\mathcal{A}}$ that still commutes to the action of the group G generated by ϕ. Thus $\tilde{f}_M$ descends to an automorphism φ_M of X. The map $M \to \varphi_M$ embeds $\mathrm{GL}(2;\Lambda)/G$ as a subgroup of $\mathrm{Aut}\,(X)$.

Let H be the group of matrices in $\mathrm{SL}(2;\Lambda)$ that are congruent to the identity matrix modulo the ideal $(1-\mathbf{j})\mathbb{Z}[\mathbf{j}]$.

Lemma 4.4. *The natural morphism* $H \to \mathrm{GL}(2;\Lambda)/G$ *is injective and its image is exactly the stabilizer of* S.

Proof. The injectivity is obvious since $G \cap \mathrm{SL}(2;\Lambda) = \mathrm{id}$. For the surjectivity, note that the automorphism f_M fixes the nine points p_i if an only if 3 divides $(2+\mathbf{j})\,(a-1)$, $(2+\mathbf{j})\,b$, $(2+\mathbf{j})\,c$ and $(2+\mathbf{j})\,(d-1)$ where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $3 = (2+\mathbf{j})\,(1-\mathbf{j})$, M belongs to the stabilizer of S if and only if its reduction modulo $(1-\mathbf{j})\mathbb{Z}[\mathbf{j}]$ is the identity matrix. Among the units of $\mathbb{Z}[\mathbf{j}]$, only 1, $\mathbf{j}$ and $\mathbf{j}^2$ are congruent to 1 modulo $1-\mathbf{j}$. Thus $\det M \in \{1, \mathbf{j}, \mathbf{j}^2\}$ and we are done. □

Let M be an element of $\mathrm{GL}(2;\Lambda)$ with eigenvalues α and β, and $V = (\Lambda^* \times \Lambda^*) \otimes_{\mathbb{Z}} \mathbb{C}$. Then the complex eigenvalues of the endomorphism $\wedge^2 M$ of the real vector space $\wedge^2_{\mathbb{R}} V$ are $|\alpha|^2$, $|\beta|^2$, $\alpha\overline{\beta}$, $\overline{\alpha}\beta$, $\alpha\beta$ and $\overline{\alpha}\overline{\beta}$. Note that for dimension reasons we have a G-equivariant exact sequence

$$0 \longrightarrow (\wedge^2_{\mathbb{R}} V)^G \longrightarrow \wedge^2_{\mathbb{R}} V \longrightarrow \wedge^2_{\mathbb{C}} V \longrightarrow 0$$

since ϕ acts by multiplication by $\mathbf{j}^2$ on $\wedge^2_{\mathbb{C}} V$. Besides f_M acts by $\det(M) = \alpha\beta$ on $\wedge^2_{\mathbb{C}} V$, so the eigenvalues of the corresponding $\mathbb{R}$-linear endormorphism are $\alpha\beta$ and $\overline{\alpha}\overline{\beta}$. Thus the eigenvalues of $\wedge^2 M$ on the subspace $(\wedge^2_{\mathbb{R}} V)^G$ are exactly $|\alpha|^2$, $|\beta|^2$, $\alpha\overline{\beta}$ and $\overline{\alpha}\beta$. We conclude that for any element M

in $\mathrm{GL}(2;\Lambda)$ with spectral radius r_M, the spectral radius of M acting on $\left(\wedge^2_{\mathbb{Z}}(\Lambda^* \times \Lambda^*)\right)^G$ is r_M^2. This means that the first dynamical degree of $\tilde{f}_M$ is given by the formula $\lambda_1(\tilde{f}_M) = r_M^2$. More precisely, the characteristic polynomial of $\tilde{f}_M^*$ acting on $\mathrm{Pic}(X)$ is

$$(x-1)^9(x-|\alpha|^2)(x-|\beta|^2)(x-\overline{\alpha}\beta)(x-\alpha\overline{\beta}).$$

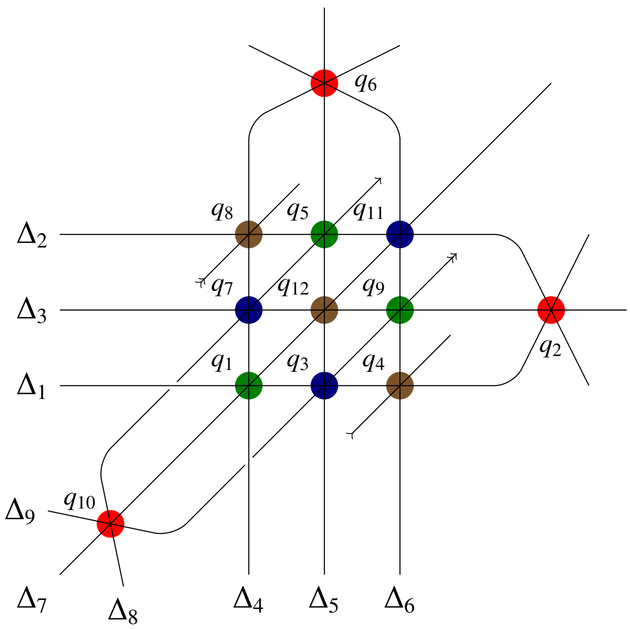
4.1.3. *Explicit realisation in the Cremona group.* According to Lemma 4.1, the Kummer surface X is obtained by blowing up $\mathbb{P}^2$ along 12 points, and an explicit morphism $p\colon X \to \mathbb{P}^2$ is obtained by blowing down the -1 curves $(\mathscr{E}_i)_{1\leq i\leq 12}$. We can therefore define for any matrix M in $\mathrm{GL}(2;\mathbb{Z}[\mathbf{j}])$ a Cremona transformation $\psi_M\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by the formula

$$\psi_M = p^{-1} \circ \varphi_M \circ p.$$

It is an interesting question to find explicit formulas for the map ψ_M. The first step to understand the maps ψ_M is to describe explicitly the configuration of lines $(p(E_i))_{1\leq i\leq 9}$ in $\mathbb{P}^2$ since the intersection points of these lines give the indeterminacy locus of p^{-1}. Let us put

$$\begin{cases} q_i = p(\mathscr{E}_i) & \text{for} \quad 1 \leq i \leq 12 \\ \Delta_j = p(E_j) & \text{for} \quad 1 \leq j \leq 9 \end{cases}$$

The two configurations $\{C_i, p_j\}$ and $\{\Delta_j, q_i\}$ are projectively dual.



We have the following result (which is almost [34, Proposition 1] with a small additional ingredient).

Proposition 4.5. *There exists linear coordinates x, y, z on $\mathbb{P}^2$ such that*

$$\begin{cases} \Delta_1 = \{y = z\} & \Delta_2 = \{y = \mathbf{j}z\} & \Delta_3 = \{y = \mathbf{j}^2 z\} & \Delta_4 = \{x = z\} & \Delta_5 = \{x = \mathbf{j}^2 z\} \\ \Delta_6 = \{x = \mathbf{j}z\} & \Delta_7 = \{y = x\} & \Delta_8 = \{y = \mathbf{j}^2 x\} & \Delta_9 = \{y = \mathbf{j}x\} \end{cases}$$

and

$$\begin{cases} q_1 = (1:1:1) & q_2 = (1:0:0) & q_3 = (\mathbf{j}^2:1:1) & q_4 = (\mathbf{j}:1:1) \\ q_5 = (\mathbf{j}^2:\mathbf{j}:1) & q_6 = (0:1:0) & q_7 = (1:\mathbf{j}^2:1) & q_8 = (1:\mathbf{j}:1) \\ q_9 = (\mathbf{j}:\mathbf{j}^2:1) & q_{10} = (0:0:1) & q_{11} = (\mathbf{j}:\mathbf{j}:1) & q_{12} = (\mathbf{j}^2:\mathbf{j}^2:1) \end{cases}$$

Proof. Since no line passes through three points in the set $\{q_1, q_2, q_6, q_{10}\}$, there exist unique linear coordinates on $\mathbb{P}^2$ such that $q_1 = (1:1:1)$, $q_2 = (1:0:0)$, $q_6 = (0:1:0)$ and $q_{10} = (0:0:1)$. Then $\Delta_1 = \{y = z\}$, $\Delta_4 = \{x = z\}$ and $\Delta_7 = \{x = y\}$. Now the line $\{z = 0\}$ (resp. $\{x = 0\}$) cannot be equal to Δ_5 or Δ_6 since it passes through q_2 (resp. q_{10}). Hence there exist nonzero complex numbers α and β such that $\Delta_5 = \{x = \alpha z\}$ and $\Delta_6 = \{x = \beta z\}$. Then $\Delta_3 = \{y = \alpha z\}$ and $\Delta_2 = \{y = \beta z\}$. Hence in the affine chart $z = 1$, $q_3 = (\alpha, 1)$, $q_4 = (\beta, 1)$, $q_7 = (1, \alpha)$, $q_{12} = (\alpha, \alpha)$, $q_9 = (\beta, \alpha)$, $q_8 = (1, \beta)$, $q_5 = (\alpha, \beta)$ and $q_{11} = (\beta, \beta)$. Using that $\{q_{10}, q_3, q_9, q_8\}$ are aligned, we get $\alpha^2 = \beta$ and $\alpha = \beta^2$ so that $\alpha \in \{\mathbf{j}, \mathbf{j}^2\}$. This gives two distinct configurations, which are not projectively isomorphic, although they are complex conjugate [13]. Using the affine coordinate x as a coordinate on Δ_1, we see that the cross-ratio $[q_1, q_2, q_3, q_4]$ is

$$[q_1, q_2, q_3, q_4] = [1, \infty, \alpha, \alpha^2] = \frac{1-\alpha}{1-\alpha^2} = \frac{1}{1+\alpha} = -\alpha.$$

But this cross ratio is easy to compute, since the points q_i, $1 \leq i \leq 4$ can be identified with the intersections of the strict transforms of the curves C_i, $1 \leq i \leq 4$ with the exceptional divisor $\widetilde{E}_1$ via the isomorphisms $\widetilde{E}_1 \simeq E_1 \simeq \Delta_1$. Hence we get

$$[q_1, q_2, q_3, q_4] = [0, \infty, 1, -\mathbf{j}] = -\mathbf{j}^2$$

so that $\alpha = \mathbf{j}^2$ and we are done. □

The next step to understand the birational maps ψ_M is to compute their degrees. Although this result is not strictly necessary for us, we include it because it can be useful for explicit computations.

Proposition 4.6. *For any matrix* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *in* $\mathrm{GL}(2; \mathbb{Z}[\mathbf{j}])$,

$$\deg \psi_M = |a+d|^2 + |c - \mathbf{j}b|^2 + \left(1 + \frac{\sqrt{3}}{2}\right) |\mathbf{i}a - \mathbf{j}^2 b - \mathbf{j}c - \mathbf{i}d|^2 + \left(1 - \frac{\sqrt{3}}{2}\right) |\mathbf{i}a + \mathbf{j}^2 b + \mathbf{j}c - \mathbf{i}d|^2 - 3$$

Proof. Let us recall the three following commutative diagrams:

$$\begin{array}{ccc} \widetilde{\mathcal{A}} & \xrightarrow{\tilde{f}_M} & \widetilde{\mathcal{A}} \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ \mathcal{A} & \xrightarrow{f_M} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} \widetilde{\mathcal{A}} & \xrightarrow{\tilde{f}_M} & \widetilde{\mathcal{A}} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{\varphi_M} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\varphi_M} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ \mathbb{P}^2 & \overset{\psi_M}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

We have $p^*\Delta_1 = E_1 + \sum_{i=1}^4 \mathscr{E}_i$. Hence, if ζ and ϵ denote the divisors $\sum_{i=1}^4 C_i$ and $\sum_{i=1}^9 \widetilde{E}_i$ on $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ respectively,

13. This point is not clearly explained in [34, Proposition 1] where the terminology "essentially unique" can be slightly misleading.

$$
\begin{aligned}
(p\circ\pi)^*\Delta_1 &= 3\widetilde{E}_1 + (\delta^* C_1 - \widetilde{E}_1 - \widetilde{E}_4 - \widetilde{E}_7) + (\delta^* C_2 - \widetilde{E}_1 - \widetilde{E}_2 - \widetilde{E}_3)\\
&\quad + (\delta^* C_3 - \widetilde{E}_1 - \widetilde{E}_5 - \widetilde{E}_9) + (\delta^* C_4 - \widetilde{E}_1 - \widetilde{E}_4 - \widetilde{E}_8)\\
&= \delta^*\zeta\ -\epsilon
\end{aligned}
$$

so that

$$
\deg\,\psi_M = p^*\Delta_1\,.\,(p\circ\varphi_M)^*\Delta_1 = \frac{(p\circ\varphi_M\circ\pi)^*\Delta_1\,.\,(p\circ\pi)^*\Delta_1}{3} = \frac{f_M^*\zeta\,.\,\zeta + \tilde{f}_M^*\epsilon\,.\,\epsilon}{3} = \frac{f_M^*\zeta\,.\,\zeta}{3} - 3.
$$

A routine computation yields the value of the cohomology classes of the curves C_i in $\mathrm{H}^2(\mathcal{A},\mathbb{C})$:

$$
\begin{aligned}
[C_1] &= \frac{\mathbf{i}\sqrt{3}}{3} dz\wedge d\bar{z}\\
[C_2] &= \frac{\mathbf{i}\sqrt{3}}{3} dw\wedge d\bar{w}\\
[C_3] &= \frac{\mathbf{i}\sqrt{3}}{3} (d\bar{z}\wedge dw - dz\wedge d\bar{w} + dz\wedge d\bar{z} + dw\wedge d\bar{w})\\
[C_4] &= \frac{\mathbf{i}\sqrt{3}}{3} \Big(\mathbf{j}dz\wedge d\bar{w} - \mathbf{j}^2 d\bar{z}\wedge dw + dz\wedge d\bar{z} + dw\wedge d\bar{w}\Big).
\end{aligned}
$$

Thus, if $\mu = (\mathbf{j}-1)$,

$$
[\zeta] = \frac{\mathbf{i}\sqrt{3}}{3}\,(3dz\wedge d\bar{z} + 3dw\wedge d\bar{w} + \mu\, dz\wedge d\bar{w} - \bar{\mu}\, d\bar{z}\wedge dw)
$$

so that

$$
\deg\psi_M = 3(|a|^2+|b|^2+|c|^2+|d|^2) + 2\,\Re\Big\{(\mathbf{j}-1)(a\bar{c}+b\bar{d}-\bar{a}b-\bar{c}d) + \mathbf{j}b\bar{c} - a\bar{d}\Big\} - 3.
$$

The result follows by a direct calculation. □

Remark 4.7.

(i) The degree of ψ_M depends only on the class of M in $\mathrm{GL}(2;\Lambda)/G$, and this can be verified directly on the explicit expression of $\deg\psi_M$ given by Proposition 4.6.

(ii) We have $\deg\psi_M = Q(a,b,c,d) - 3$ where Q is a positive-definite hermitian form. Hence for any positive integer N, the set

$$
\{M\in \mathrm{GL}(2;\mathbb{Z}[\mathbf{j}])\ \text{s.t.}\ \deg\Psi_M \le N\}
$$

is finite.

Theorem 4.8. *The matrices* $M_1 = \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}$, $M_2 = \begin{pmatrix}0 & 1\\ -\mathbf{j} & 0\end{pmatrix}$ *and* $M_3 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ *generate* $\mathrm{GL}(2;\mathbb{Z}[\mathbf{j}])$ *as a semigroup. Besides, the corresponding Cremona transformations have the following explicit description*:

$$
\begin{cases}
\psi_{M_1}: (x:y:z)\to(x+y+z : \mathbf{j}x+y+\mathbf{j}^2 z : \mathbf{j}x+\mathbf{j}^2 y+z)\\
\psi_{M_2}: (x:y:z)\to(x+y+z : x+\mathbf{j}^2 y+\mathbf{j}z : x+\mathbf{j}y+\mathbf{j}^2 z)\\
\psi_{M_3}: (x:y:z)\to(a(x:y:z), b(x:y:z), c(x:y:z))
\end{cases}
$$

where

$$\begin{cases} a(x:y:z) = x^2+y^2+z^2-\mathbf{j}^2(xy+xz+yz) \\ b(x:y:z) = x^2+\mathbf{j}y^2+\mathbf{j}^2z^2-\mathbf{j}xy-xz-\mathbf{j}^2yz \\ c(x:y:z) = x^2+\mathbf{j}^2y^2+\mathbf{j}z^2-xy-\mathbf{j}xz-\mathbf{j}^2yz \end{cases}$$

Proof. The ring $\mathbb{Z}[\mathbf{j}]$ being euclidean, any matrix in $\mathrm{GL}(2;\mathbb{Z}[\mathbf{j}])$ can be put in diagonal form after performing elementary row and column operations, corresponding to multiplications on the right and on the left by matrices of the type $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$. First note that the diagonal matrices can be expressed using M_1, M_2 and M_3: we have

$$\begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{j} \end{pmatrix} = M_2M_3 \qquad \text{and} \qquad M_3M_2M_3^2 = \begin{pmatrix} -\mathbf{j} & 0 \\ 0 & 1 \end{pmatrix}$$

and these two matrices generate all invertible diagonal matrices, since $\mathbb{Z}[\mathbf{j}]^\times$ is the cyclic subgroup of $\mathbb{C}^\times$ generated by $-\mathbf{j}$. We now deal with the transvection matrices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times M_3M_1(M_2M_3)^3$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \times M_1(M_2M_3)^3M_3.$$

$$\begin{pmatrix} 1 & \mathbf{j} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{j}^2 & 0 \\ 0 & \mathbf{j} \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j}^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \mathbf{j} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j}^2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{j}^2 & 0 \\ 0 & \mathbf{j} \end{pmatrix}$$

Hence M_1, M_2 and M_3 generate $\mathrm{GL}(2;\mathbb{Z}[\mathbf{j}])$ as a group. Since these matrices are of finite order, they also generate $\mathrm{GL}(2;\mathbb{Z}[\mathbf{j}])$ as a semigroup.

It remains to compute the explicit expressions of the ψ_{M_i}, $1 \leq i \leq 3$. First it is easy to see that ψ_{M_1} and ψ_{M_2} have no indeterminacy point, because the maps φ_{M_1} and φ_{M_2} preserve globally the twelve exceptional curves $\mathscr{E}_j$, $1 \leq j \leq 12$. Hence ψ_{M_1} and ψ_{M_2} are linear (this could also be checked using Proposition 4.6), and therefore entirely determined by its action on the points q_k. In order to compute ψ_{M_3}, we first note that it is a quadratic involution. Indeed, φ_{M_3} leaves globally invariant the set $\{\mathscr{E}_1, \mathscr{E}_5, \mathscr{E}_9, \mathscr{E}_2, \mathscr{E}_6, \mathscr{E}_{10}, \mathscr{E}_3, \mathscr{E}_7, \mathscr{E}_{11}\}$, so that the indeterminacy locus of ψ_{M_3} consists of simple points in the list $\{q_4, q_8, q_{12}\}$. Since ψ_{M_3} is not linear (otherwise all ψ_M would also be linear), it must be a quadratic involution whose indeterminacy locus is the set $\{q_4, q_8, q_{12}\}$. It is completely determined by its action on the remaining points points q_k for $k \notin \{4, 8, 12\}$. □

4.2. **Action of the automorphism group on infinitesimal deformations.** In this section, we will present two different approaches to prove the following theorem:

Theorem 4.9. *Let X be the rational Kummer surface associated with the lattice* $\mathbb{Z}[\mathbf{j}]$ *of Eisenstein integers. For any element M in the group* $\mathrm{GL}(2;\mathbb{Z}[\mathbf{j}])$*, let* σ_M *be the permutation of S given by the action of* f_M*, and let* $\mathcal{P}_M$ *be the set of the* 9 *eigenvalues of the permutation matrix associated with*

σ_M. *Then the characteristic polynomial* Q_M *of the endomorphism* φ_M^* *of* $\mathrm{H}^1(X, \mathrm{T}X)$ *is given by the formula*

$$Q_M(x) = \prod_{\lambda \in \mathcal{P}_M \setminus \{1\}} \left(x - \frac{\lambda}{\alpha\beta^2}\right)\left(x - \frac{\lambda}{\alpha^2\beta}\right)$$

Remarks 4.10.

(i) We have $\#\mathcal{P}_M = 9$, so that $\deg Q_M = 16 = 2 \times 12 - 8$.

(ii) If M is in the group G generated by ϕ, then $Q_M(x) = (x-1)^{16}$ as expected. Indeed, if $M = \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix}$ then $\alpha = \beta = \mathbf{j}$ and $\mathcal{P}_M = \{1, 1, 1, 1, 1, 1\}$.

Corollary 4.11.

(i) If M *is a matrix in* H*, then* $Q_M(x) = (x-\alpha)^8(x-\beta)^8$.

(ii) If M *is not of finite order in* $\mathrm{GL}(2; \mathbb{Z}[\mathbf{j}])/\langle \mathbf{j}\,\mathrm{id} \rangle$*, then* φ_M *is rigid.*

The first approach of the proof of Theorem 4.9 will use the Atiyah-Bott fixed point theorem to compute the trace of the action of φ_M^* on $\mathrm{H}^1(X, \mathrm{T}X)$ for M in H. Knowing this for all iterates of M, we obtain all eigenvalues of $\tilde{f}_M$. We will limit ourselves to the case where M is in H (that is to the statement (*i*) of Corollary 4.11), but all other cases can be dealt with using the same method. Since these calculations don't bring anything new conceptually (and since the second proof is valid for any M in $\mathrm{GL}(2; \Lambda)$) we omit them. The other approach is to the machinery of sheaves: the tangent bundles $\mathrm{T}\mathcal{A}$, $\mathrm{T}\widetilde{\mathcal{A}}$ and $\mathrm{T}X$ are related by some exact sequences, allowing to compare their respective cohomology groups.

4.2.1. *First proof by the Atiyah-Bott formula.* Let us divide the set of fixed points of $\tilde{f}_M$ into two parts: the first part Θ_1 consists of fixed points outside the exceptional divisors E_i; and the second part Θ_2 consists of the remaining fixed points.

Proposition 4.12. *Let* M *be a matrix in* H *such that* $M^3 \neq \mathrm{id}$. *Then the automorphism* $\tilde{f}_M$ *has* $|\mathrm{Tr}(M)|^2 + 11$ *fixed points on the Kummer surface* X.

Proof. The first step consists in proving that the fixed points of $\tilde{f}_M$ are non-degenerate, *i.e.* that 1 is never an eigenvalue of the differential of $\tilde{f}_M$ at a fixed point. We deal with fixed points in Θ_1 and Θ_2 separately.

The points of Θ_1 correspond to points p in $A \setminus S$ such that $f_M(p)$ lies in the orbit $G \cdot p$, modulo the action of G. For any such point, the differential of $\tilde{f}_M$ identifies with M, $\mathbf{j}M$ or $\mathbf{j}^2 M$, so that 1 is never an eigenvalue.

The set Θ_2 can be described as follows: the automorphism $\tilde{f}_M$ acts by the projective transformation $\mathbb{P}(M)$ on the curve E_i via the identification

$$E_i \simeq \widetilde{E}_i \simeq \mathbb{P}(\mathrm{T}_{p_i} V) \simeq \mathbb{P}(V).$$

Therefore we get two fixed points q_α and q_β on each E_i corresponding to the eigenspaces of M. This gives a concrete description of Θ_2, which consists of 18 points (two distinct points on each E_i). To compute the differential of $\tilde{f}_M$ at such a fixed point, we can assume without loss of generality that this point lies in E_1. Let (e, f) be an eigenvector of M for the eigenvalue α and assume for simplicity that $e \neq 0$. If (x, y) are the standard coordinates on V, we define coordinates (u, v) in $\widetilde{\mathscr{A}}$ near $\widetilde{E}_1$ by putting $x = u$ and $y = uv$. Then if we set $Z = u^3$ and $T = v$, (Z, T) are holomorphic coordinates on X near q_α. In these coordinates, $q_\alpha = \left(0, \dfrac{f}{e}\right)$ and $\tilde{f}_M$ is given by

$$(Z, T) \mapsto \left(Z(a + bT)^3, \frac{c + dT}{a + bT}\right).$$

so that the eigenvalues of $\mathrm{d}\tilde{f}_M$ at q_α are α^3 and α^{-2}, which are different from 1.

We can now apply the Lefschetz fixed point formula. As $\mathrm{H}^1(X, \mathbb{R}) = \mathrm{H}^3(X, \mathbb{R}) = 0$, we get that the number of fixed points of $\tilde{f}_M$ is equal to $\mathrm{Tr}\ \widetilde{f}^{\,*}_{M\,|\mathrm{H}^2(X,\mathbb{R})} + 2$. Then we can use formula (4.2): the trace on the part $\left(\wedge^2_{\mathbb{Z}}(\Lambda^* \times \Lambda^*)\right)^G$ is the sum

$$|\alpha|^2 + |\beta|^2 + \alpha\overline{\beta} + \overline{\alpha}\beta = |\alpha + \beta|^2;$$

and since M belongs to H, $\tilde{f}_M$ acts trivially on the factor $\bigoplus_{i=1}^9 \mathbb{Z}[\widetilde{E}_i]$ so that the trace on this factor is 9. This gives the required result. □

We will now study in greater details the set Θ_1. Let us define three subsets S_1, S_2, S_3 of A as follows:

$$S_1 = \{p \in A \smallsetminus S \ \mid f_M(p) = p\}, S_2 = \{p \in A \smallsetminus S \ \mid f_M(p) = \phi(p)\}, S_3 = \{p \in A \smallsetminus S \ \mid f_M(p) = \phi^2(p)\}.$$

To compute the cardinality of the S_i's, we use the following statement:

Lemma 4.13. *For any matrix P in $\mathrm{GL}(2; \Lambda)$ such that 1 is not an eigenvalue of P, the automorphism f_P of the complex torus A has $\left|1 - \mathrm{Tr}(P) + \det P\right|^2$ fixed points.*

Proof. Since the fixed points of f_P are non-degenerate, we can apply the holomorphic Lefschetz fixed point formula ([22, p. 426], [36], [10]); this gives

$$1 - \overline{\mathrm{Tr}(P)} + \overline{\det P} = \#\,\mathrm{Fix}(f_P) \times \frac{1}{1 - \mathrm{Tr}(P) + \det P}.$$

□

Corollary 4.14. *Let M be a matrix in H such that $M^3 \neq \mathrm{id}$. One has the following equalities:*

$$\#S_1 = \left|\mathrm{Tr}(M) - 2\right|^2 - 9 \qquad \textit{and} \qquad \#S_2 = \#S_3 = \left|\mathrm{Tr}(M) + 1\right|^2 - 9.$$

Proof. This follows directly from Lemma 4.13, since

$$S_1 = \mathrm{Fix}(f_M) \smallsetminus S, \qquad S_2 = \mathrm{Fix}(f_{\mathbf{j}^2 M}) \smallsetminus S \qquad \text{and} \qquad S_3 = \mathrm{Fix}(f_{\mathbf{j}M}) \smallsetminus S.$$

□

The group G acts transitively on each set S_i. Then Θ_1 can be written as the disjoint union of the quotients S_i/G. Remark that we can obtain in this way the result of Proposition 4.12: indeed, if $t = \mathrm{Tr}\,(M)$,

$$\#\,\Theta_1 = \frac{\left|t-2\right|^2}{3} - 3 + 2\left(\frac{\left|1+t\right|^2}{3} - 3\right) = \left|t\right|^2 - 7$$

Since $\#\,\Theta_2 = 18$, this gives $\#\,\mathrm{Fix}\,(\tilde{f}_M) = \left|t\right|^2 + 11$.

As $\mathrm{H}^0(X, \mathrm{T}X) = \mathrm{H}^2(X, \mathrm{T}X) = 0$, the holomorphic Atiyah-Bott fixed point formula [1, Thm. 4.12] yields the following result :

Proposition 4.15. *The trace of $\tilde{f}_M^*$ acting on $\mathrm{H}^1(X, \mathrm{T}X)$ is equal to the sum $-\sum_x \dfrac{\mathrm{Tr}\ \mathrm{d}f_x^{-1}}{\det(\mathrm{id} - \mathrm{d}f_x)}$ where x runs through the fixed points of $\tilde{f}_M$.*

Corollary 4.16. *Let M be a matrix in H such that $M^3 \neq \mathrm{id}$. Then the trace of $\tilde{f}_M^*$ acting on $\mathrm{H}^1(X, \mathrm{T}X)$ is* $8\,\mathrm{Tr}\,(M)$.

Proof. We divide the fixed point set of $\tilde{f}_M$ in four parts: S_1/G, S_2/G, S_3/G and Θ_2. The first three parts correspond to Θ_1. We start by computing the contribution of Θ_2 in the sum of Proposition 4.15. Let us put $t = \mathrm{Tr}\,(M) = \alpha + \alpha^{-1}$. Any pair of fixed points in a divisor E_i yields the term

$$\frac{\alpha^{-3} + \alpha^2}{(1-\alpha^3)(1-\alpha^{-2})} + \frac{\alpha^3 + \alpha^{-2}}{(1-\alpha^{-3})(1-\alpha^2)}$$

so that the contribution of Θ_2 is $-9t - \dfrac{6}{t-2} - \dfrac{3}{1+t}\cdot$ Now the contributions of S_1/G, S_2/G and S_3/G are respectively

$$\left(\frac{\left|t-2\right|^2}{3} - 3\right)\frac{t}{2-t}, \qquad \left(\frac{\left|1+t\right|^2}{3} - 3\right)\frac{-\mathbf{j}^2\, t}{1+t} \qquad \text{and} \qquad \left(\frac{\left|1+t\right|^2}{3} - 3\right)\frac{-\mathbf{j}\, t}{1+t}$$

so that the contribution of Θ_1 is $t + \dfrac{3t}{t-2} - \dfrac{3t}{1+t}\cdot$ Adding the contributions of Θ_1 and Θ_2 we get $-8t$. □

4.2.2. *Second proof by sheaf theory.* We start by fixing some conventions and notations. For any group G and any vector space W, a G-module means a *contravariant* representation of G in $\mathrm{GL}(W)$. We denote the vector space $\mathbb{C}^2$ by V. If nothing else is specified, we consider V as a natural $\mathrm{GL}(2;\Lambda)$-module, where a matrix M acts by M^{-1}. We start by the two following exact sequences of sheaves:

First exact sequence of sheaves on $\widetilde{\mathcal{A}}$ relating TA *and* $\mathrm{T}\widetilde{\mathcal{A}}$ *via the blowup map δ*

$$0 \longrightarrow \mathrm{T}\widetilde{\mathcal{A}} \longrightarrow \delta^*\mathrm{T}\mathcal{A} \longrightarrow \bigoplus_{i=1}^{9} \iota_{E_i *}\mathrm{T}_{\widetilde{E}_i}(-1) \longrightarrow 0. \tag{4.3}$$

This sequence is $\mathrm{GL}(2;\Lambda)$-equivariant and G-equivariant. Let us explain its construction: the first map is the differential of the map $\delta\colon \widetilde{\mathcal{A}} \to \mathcal{A}$ corresponding to the blowup of the nine points in S; as a sheaf morphism it is injective. For any i with $1 \leq i \leq 9$ and any point $[\ell]$ in E_i (corresponding to a line ℓ in $\mathrm{T}_{p_i}\mathcal{A}$), the image of $(\delta_*)_{[\ell]}\colon \mathrm{T}_{[\ell]}\widetilde{\mathcal{A}} \to \mathrm{T}_{p_i}\mathcal{A}$ is precisely the line ℓ. Thanks to the Euler exact sequence [30, Prop. 2.4.4]

$$0 \longrightarrow \mathcal{O}_{E_i}(-1) \longrightarrow \mathrm{T}_{p_i}\mathcal{A} \otimes \mathcal{O}_{E_i} \longrightarrow \mathrm{T}E_i(-1) \longrightarrow 0 \tag{4.4}$$

the complex line $\frac{\mathrm{T}_{p_i}\mathcal{A}}{\mathrm{Im}\,(\delta_*)_{[\ell]}}$ identifies canonically with the fiber of $\mathrm{T}_{\widetilde{E}_i}(-1)$ at $[\ell]$.

Remark that $\mathrm{T}_{\widetilde{E}_i}(-1)$ is isomorphic to $\mathcal{O}_{E_i}(1)$, but this isomorphism is not canonical and cannot be made compatible in any way with the action of $\mathrm{GL}(2;\Lambda)$.

Second exact sequence of sheaves on X relating $\mathrm{T}\widetilde{\mathcal{A}}$ and $\mathrm{T}X$ via the cyclic cover π

$$0 \longrightarrow \pi_*(\mathrm{T}\widetilde{\mathcal{A}})^G \longrightarrow \mathrm{T}X \longrightarrow \bigoplus_{i=1}^{9} \iota_{E_i *}\mathrm{N}_{E_i/X} \longrightarrow 0. \tag{4.5}$$

This sequence is $\mathrm{GL}(2;\Lambda)$-equivariant. Let us again explain its construction: for any open subset U of X, the sections of the sheaf $\pi_*(\mathrm{T}\widetilde{\mathcal{A}})^G$ on U are exactly the G-invariant holomorphic vector fields on $\pi^{-1}(U)$. We can take holomorphic coordinates (z, w) and (x, y) near a point of $\widetilde{E}_i$ and its image in X such that $\pi(z, w) = (z^3, w)$, $\phi(z, w) = (\mathbf{j}z, w)$ and $\widetilde{E}_i = \{z = 0\}$. Therefore a G-invariant holomorphic vector field is of the form $z\alpha(z^3, w)\partial_z + \beta(z^3, w)\partial_w$, which is (outside E_i) the pull-back of the holomorphic vector field $3x\,\alpha(x, y)\partial_x + \beta(x, y)\partial_y$. The latter holomorphic vector field extends uniquely across E_i, and the extension at E_i is tangent to E_i. Conversely, the same calculation shows that every such holomorphic vector field yields a G invariant holomorphic vector field upstairs. To conclude, it suffices to note that a holomorphic vector field on an open subset of X is tangent to E_i if and only if its restriction to E_i maps to zero via the morphism $\mathrm{T}X_{E_i} \to \mathrm{N}_{E_i/X}$.

Let us introduce some notation. First we consider the natural representation Z of the symmetric group $\mathfrak{S}_9$ in $\mathbb{C}^9$. There is a natural group morphism $\mathrm{GL}(2;\Lambda) \to \mathfrak{S}_9$ given by the action on the set $S = \{p_1, \ldots, p_9\}$, so that we will consider Z as a $\mathrm{GL}(2;\Lambda)$-module. As an H-module, Z is the sum of nine copies of the trivial representation.

Let us denote by $\mathcal{F}$ the sheaf $\bigoplus_{i=1}^{9} \iota_{E_i *}\mathrm{T}_{\widetilde{E}_i}(-1)$.

Lemma 4.17. *The following assertions hold* :

(a) For any integer i with $0 \leq i \leq 2$, *the natural morphisms*

$$\begin{cases} \mathrm{H}^i(\mathcal{A}, \mathrm{T}\mathcal{A}) \longrightarrow \mathrm{H}^i(\widetilde{\mathcal{A}}, \delta^*\mathrm{T}\mathcal{A}) \\ \mathrm{H}^i(X, \pi_*\mathrm{T}\widetilde{\mathcal{A}}) \longrightarrow \mathrm{H}^i(\widetilde{\mathcal{A}}, \pi^*\pi_*\mathrm{T}\widetilde{\mathcal{A}}) \longrightarrow \mathrm{H}^i(\widetilde{\mathcal{A}}, \mathrm{T}\widetilde{\mathcal{A}}) \end{cases}$$

are $\mathrm{GL}(2;\Lambda)$*-equivariant isomorphisms.*

(b) The cohomology groups $\mathrm{H}^0(\widetilde{\mathcal{A}}, \mathcal{F})^G$ *and* $\mathrm{H}^1(\widetilde{\mathcal{A}}, \mathcal{F})^G$ *vanish.*

(c) The cohomology group $\mathrm{H}^1(X, \pi_*(\mathrm{T}\widetilde{\mathcal{A}})^G)$ *vanishes. Besides,* $\mathrm{H}^2(X, \pi_*(\mathrm{T}\widetilde{\mathcal{A}})^G)$ *is isomorphic to* $\det V \otimes V$ *as a* $\mathrm{GL}(2;\Lambda)$*-module.*

(d) For $1 \leq i \leq 9$, $\oplus_{i=1}^{9} \mathrm{H}^1(E_i, \mathrm{N}_{E_i/X})$ *is isomorphic to* $(\det V \otimes V) \otimes Z$ *as a* $\mathrm{GL}(2;\Lambda)$*-module.*

Proof. (a) We write the Leray spectral sequence for the pair $(\delta^* \mathrm{T}\mathcal{A}, \mathcal{A})$: we have

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathcal{A}, \mathrm{R}^q\delta_* \mathcal{O}_{\widetilde{\mathcal{A}}} \otimes \mathrm{T}\mathcal{A}) \qquad \text{and} \qquad \mathrm{E}_\infty^{p,q} = \mathrm{Gr}^p \mathrm{H}^{p+q}(\widetilde{\mathcal{A}}, \delta^* \mathrm{T}\mathcal{A}).$$

Since δ is the projection of a point blowup, it is known that $\mathrm{R}^q\delta_* \mathcal{O}_{\widetilde{\mathcal{A}}}$ vanishes for $q > 0$ so that the spectral sequence degenerates and we get $\mathrm{H}^p(\mathcal{A}, \mathrm{T}\mathcal{A}) = \mathrm{E}_2^{p,0} \simeq \mathrm{E}_\infty^{p,0} = \mathrm{H}^p(\widetilde{\mathcal{A}}, \delta^* \mathrm{T}\mathcal{A})$. The argument is the same for the second morphism: π being finite, $\mathrm{R}^q\pi_* \mathrm{T}\widetilde{\mathcal{A}} = 0$ for $q > 0$.

(b) The vanishing of $\mathrm{H}^1(\widetilde{\mathcal{A}}, \mathcal{F})^G$ is straightforward as $\mathrm{H}^1(\widetilde{\mathcal{A}}, \mathcal{F}) = 0$. We see that $\mathrm{H}^0(\widetilde{\mathcal{A}}, \mathcal{F})$ is isomorphic as a G-module to the direct sum $\bigoplus_{i=1}^{9} \mathrm{T}_{p_i} A$, where ϕ acts by the inverse of its differential at each point p_i. Therefore, $\mathrm{H}^0(\widetilde{\mathcal{A}}, \mathcal{F}) \simeq V^9$ and the result follows since $V^G = \{0\}$.

(c) According to *(a)* and since G is finite, we have isomorphisms

$$\mathrm{H}^i(X, \pi_*(\mathrm{T}\widetilde{\mathcal{A}})^G) \simeq \mathrm{H}^i(X, \pi_* \mathrm{T}\widetilde{\mathcal{A}})^G \simeq \mathrm{H}^i(\widetilde{\mathcal{A}}, \mathrm{T}\widetilde{\mathcal{A}})^G.$$

Using (4.3) combined with *(b)* and *(a)*, we obtain $\mathrm{GL}(2;\Lambda)$-equivariant isomorphisms $\mathrm{H}^i(\widetilde{\mathcal{A}}, \mathrm{T}\widetilde{\mathcal{A}})^G \simeq \mathrm{H}^i(\mathcal{A}, \mathrm{T}\mathcal{A})^G$ for $i = 1, 2$. Now $\mathrm{H}^i(\mathcal{A}, \mathrm{T}\mathcal{A})$ is isomorphic to $V^* \otimes V$ (resp. $\wedge^2 V^* \otimes V$) for $i = 1$ (resp. $i = 2$), where a matrix M in $\mathrm{GL}(2;\Lambda)$ acts by ${}^t\overline{M} \otimes M^{-1}$ (resp. $\wedge^2({}^t\overline{M}) \otimes M^{-1}$). This proves that $\mathrm{H}^1(\mathcal{A}, \mathrm{T}\mathcal{A})^G$ vanishes and that

$$\mathrm{H}^2(\mathcal{A}, \mathrm{T}\mathcal{A})^G = \mathrm{H}^2(\mathcal{A}, \mathrm{T}\mathcal{A}) \simeq \det V \otimes V$$

since $\overline{\mathbf{j}}^2 \mathbf{j}^{-1} = 1$. This yields the result.

(d) Taking the determinant of the Euler exact sequence (4.4), we obtain that the sheaves $\oplus_{i=1}^{9} \mathrm{K}_{\widetilde{E}_i}$ and $\oplus_{i=1}^{9} \det \mathrm{T}_{p_i}^* \mathcal{A} \otimes_{\mathbb{C}} \mathcal{O}_{\widetilde{E}_i}(-2)$ are naturally isomorphic, and this isomorphism is compatible with the action of $\mathrm{GL}(2;\Lambda)$. Since $\pi^* \mathrm{N}_{E_i/X}$ is canonically isomorphic to $\mathcal{O}_{\widetilde{E}_i}(-3)$, we get by Serre duality a chain of $\mathrm{GL}(2;\Lambda)$-equivariant isomorphisms

$$\bigoplus_{i=1}^{9} \mathrm{H}^1(E_i, \mathrm{N}_{E_i/X}) \simeq \bigoplus_{i=1}^{9} \mathrm{H}^0(\widetilde{E}_i, \mathcal{O}_{\widetilde{E}_i}(3) \otimes \mathrm{K}_{\widetilde{E}_i})^* \simeq \bigoplus_{i=1}^{9} \det \mathrm{T}_{p_i}\mathcal{A} \otimes \mathrm{H}^0(\widetilde{E}_i, \mathcal{O}_{\widetilde{E}_i}(1))^* \simeq (\det V \otimes V) \otimes Z.$$

□

We can now prove Corollary 4.11. Using the exact sequence (4.5), Lemma 4.17 *(c)*, *(d)* and the fact that $\mathrm{H}^2(X, \mathrm{T}X)$ vanishes, we get an exact sequence of H-modules

$$0 \longrightarrow \mathrm{H}^1(X, \mathrm{T}X) \longrightarrow V^9 \longrightarrow V \longrightarrow 0.$$

The result follows.

4.3. **The case of the square lattice.** For the sake of completeness and also because it is an interesting case, we also provide briefly the corresponding results for square lattices (that is Λ is the ring of Gauß integers $\mathbb{Z}[\mathbf{i}]$, E is the elliptic curve $\mathbb{C}/\Lambda$, $\mathcal{A} = E \times E$, $\phi(z, w) = (\mathbf{i}z, \mathbf{i}w)$ and G is the group of order 4 generated by ϕ). The same strategy works but the results are slightly different. In the square case, the group G is of order 4, the map ϕ (which is this case the multiplication by $\mathbf{i}$) has 4 fixed points and 12 new other points are fixed by ϕ^2. More precisely, if we put

$$\begin{cases} p_1 = (0,0) & p_2 = \left(0, \dfrac{1+\mathbf{i}}{2}\right) & p_3 = \left(\dfrac{1+\mathbf{i}}{2}, 0\right) & p_4 = \left(\dfrac{1+\mathbf{i}}{2}, \dfrac{1+\mathbf{i}}{2}\right) \\ p_5 = \left(0, \dfrac{1}{2}\right) & p_6 = \left(\dfrac{1}{2}, 0\right) & p_7 = \left(\dfrac{1}{2}, \dfrac{1+\mathbf{i}}{2}\right) & p_8 = \left(\dfrac{1+\mathbf{i}}{2}, \dfrac{1}{2}\right) \\ p_9 = \left(\dfrac{1}{2}, \dfrac{1}{2}\right) & p_{10} = \left(\dfrac{1}{2}, \dfrac{\mathbf{i}}{2}\right) & p'_5 = \left(0, \dfrac{\mathbf{i}}{2}\right) & p'_6 = \left(\dfrac{\mathbf{i}}{2}, 0\right) \\ p'_7 = \left(\dfrac{\mathbf{i}}{2}, \dfrac{1+\mathbf{i}}{2}\right) & p'_8 = \left(\dfrac{1+\mathbf{i}}{2}, \dfrac{\mathbf{i}}{2}\right) & p'_9 = \left(\dfrac{\mathbf{i}}{2}, \dfrac{\mathbf{i}}{2}\right) & p'_{10} = \left(\dfrac{\mathbf{i}}{2}, \dfrac{1}{2}\right) \end{cases}$$

then $\mathrm{Fix}\,(\phi) = \{p_1, p_2, p_3, p_4\}$, $\mathrm{Fix}\,(\phi^2) \setminus \mathrm{Fix}\,(\phi) = \{p_5, \dots, p_{10}, p'_5, \dots, p'_{10}\}$, and $p'_i = \phi(p_i)$ for $5 \leq i \leq 10$. We put $S' = \mathrm{Fix}\,(\phi)$, $S'' = \mathrm{Fix}\,(\phi^2) \setminus \mathrm{Fix}\,(\phi)$ and $S = S' \cup S''$. We also denote by Z' (resp. Z'', resp. Z) the natural representation of the symmetric group $\mathfrak{S}_4$ (resp. $\mathfrak{S}_{12}$, resp. $\mathfrak{S}_6$) in $\mathbb{C}^4$ (resp. $\mathbb{C}^{12}$, resp. $\mathbb{C}^6$). There is a natural group morphism $\mathrm{GL}(2;\Lambda) \to \mathfrak{S}_4$ (resp. $\mathrm{GL}(2;\Lambda) \to \mathfrak{S}_{12}$) given by the action on the set S' (resp. S''), so that we will consider Z' and Z'' as $\mathrm{GL}(2;\Lambda)$-modules. Note that $\mathrm{GL}(2;\Lambda)$ acts on the set of pairs $\{p_i, p'_i\}$, so that the morphism $\mathrm{GL}(2;\Lambda) \to \mathfrak{S}_{12}$ factors as

$$\mathrm{GL}(2;\Lambda) \xrightarrow{(\xi,\eta)} \mathfrak{S}_6 \times (\mathbb{Z}/2\mathbb{Z})^6 \to \mathfrak{S}_{12}$$

where the factor $\mathfrak{S}_6$ corresponds to the action on the set of pairs, and each factor in $\mathfrak{S}_2$ corresponds to the action on the corresponding pair. We consider Z as a $\mathrm{GL}(2;\Lambda)$-module via the representation ρ given as follows: for $1 \leq i \leq 6$,

$$\rho(M).\mathbf{e}_i = (\eta(M))_i \, \mathbf{e}_{\xi(M)(i)}. \tag{4.6}$$

As before, let $\widetilde{\mathcal{A}}$ be the blowup of $\mathcal{A}$ along the 16 points of S, and denote by $\widetilde{E}_1, \dots, \widetilde{E}_4, \widetilde{E}_5, \dots, \widetilde{E}_{10}, \widetilde{E}'_5, \dots, \widetilde{E}'_{10}$ the corresponding exceptional divisors. The quotient X/G is a basic rational surface that can be obtained by blowing up $\mathbb{P}^2$ in 13 points. We consider again the diagram

$$\begin{array}{ccc} & \widetilde{\mathcal{A}} & \\ {}^{\delta}\swarrow & & \searrow^{\pi} \\ A & & X \end{array}$$

If we put $E_i = \pi(\widetilde{E}_i)$, then π is the cyclic covering of order 4 along the divisor $\sum_{i=1}^{4} 4E_i + \sum_{i=5}^{10} 2E_i$. In particular,

$$\pi^*(E_i) = \begin{cases} 4\widetilde{E}_i & \text{if } \; 1 \leq i \leq 4 \\ 2\widetilde{E}_i + 2\widetilde{E}'_i & \text{if } \; 5 \leq i \leq 10 \end{cases}$$

Theorem 4.18. *Let X be the rational Kummer surface associated with the lattice $\mathbb{Z}[\mathbf{i}]$ of Gauß integers. For any element M in $\mathrm{GL}(2;\mathbb{Z}[\mathbf{i}])$, let $\mathcal{P}_M$ be the set of eigenvalues of $\rho(M)$, let σ'_M be the permutation of S' given by the action of f_M, and let $\mathcal{P}'_M$ be the set of 4 eigenvalues of the permutation* *matrix associated with σ'_M. Then the characteristic polynomial Q_M of the endomorphism ${\varphi_M}^*$ of* $\mathrm{H}^1(X, \mathrm{T}X)$ *is given by the formula*

$$Q_M(x) = \prod_{\lambda \in \mathcal{P}_M} \left(x - \frac{\lambda}{\alpha\beta}\right) \prod_{\mu \in \mathcal{P}'_M} \left\{ \left(x - \frac{\mu}{\alpha^3\beta}\right)\left(x - \frac{\mu}{\alpha\beta^3}\right)\left(x - \frac{\mu}{\alpha^2\beta^2}\right) \right\}.$$

Remarks 4.19.

(i) We have $\#\mathcal{P}_M = 6$ and $\#\mathcal{P}'_M = 4$ so that $\deg Q_M = 18 = 2 \times 13 - 8$.

(ii) If M is in the group G generated by the automorphism ϕ, then $Q_M(x) = (x-1)^{18}$ as expected. Indeed, if $M = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}$ then $\alpha = \beta = \mathbf{i}$, $\mathcal{P}_M = \{-1, -1, -1, -1, -1, -1\}$ and $\mathcal{P}'_M = \{1, 1, 1, 1\}$.

Proof. We follow the strategy developed for hexagonal lattices in §4.2.2. Coming back to Lemma 4.17, the main difference is that $\mathrm{H}^2(\mathcal{A}, \mathrm{T}\mathcal{A})^G = 0$ since $\bar{\mathbf{i}}^2\, \mathbf{i}^{-1} = \mathbf{i} \neq 1$. Hence we get an isomorphism of $\mathrm{GL}(2;\Lambda)$-modules

$$\mathrm{H}^1(X, \mathrm{T}X) \simeq \bigoplus_{i=1}^{10} \mathrm{H}^1(E_i, \mathrm{N}_{E_i/X}).$$

The right-hand side of the previous isomorphism splits as the direct sum of two $\mathrm{GL}(2;\Lambda)$-modules: $\bigoplus_{i=1}^{5} \mathrm{H}^1(E_i, \mathrm{N}_{E_i/X})$ and $\bigoplus_{i=5}^{10} \mathrm{H}^1(E_i, \mathrm{N}_{E_i/X})$. Besides, we have

$$\begin{cases} \pi^*\mathrm{N}_{E_i/X} \simeq \mathcal{O}_{\widetilde{E}_i}(-4) & \text{if} \quad 1 \leq i \leq 4 \\ \pi^*\mathrm{N}_{E_i/X} \simeq \mathcal{O}_{\widetilde{E}_i}(-2) \oplus \mathcal{O}_{\widetilde{E}'_i}(-2) & \text{if} \quad 5 \leq i \leq 10 \end{cases}$$

Using Serre duality, we have $\mathrm{GL}(2;\Lambda)$-equivariant isomorphisms

$$\begin{aligned} \bigoplus_{i=1}^{4} \mathrm{H}^1(E_i, \mathrm{N}_{E_i/X}) &\simeq \bigoplus_{i=1}^{4} \mathrm{H}^0(\widetilde{E}_i, \mathcal{O}_{\widetilde{E}_i}(4) \otimes \mathrm{K}_{\widetilde{E}_i})^* \simeq \bigoplus_{i=1}^{4} \det \mathrm{T}_{p_i}\mathcal{A} \otimes \mathrm{H}^0(\widetilde{E}_i, \mathcal{O}_{\widetilde{E}_i}(2))^* \\ &\simeq (\det V \otimes \mathrm{Sym}^2\, V) \otimes Z' \end{aligned}$$

and

$$\begin{aligned} \bigoplus_{i=5}^{10} \mathrm{H}^1(E_i, \mathrm{N}_{E_i/X}) &\simeq \left(\bigoplus_{i=5}^{10} \left\{ \mathrm{H}^0(\widetilde{E}_i, \mathcal{O}_{\widetilde{E}_i}(2) \otimes \mathrm{K}_{\widetilde{E}_i})^* \oplus \mathrm{H}^0(\widetilde{E}'_i, \mathcal{O}_{\widetilde{E}'_i}(2) \otimes \mathrm{K}_{\widetilde{E}'_i})^* \right\} \right)^G \\ &\simeq \left(\bigoplus_{i=5}^{10} \left\{ \det \mathrm{T}_{p_i}\mathcal{A} \otimes \mathrm{H}^0(\widetilde{E}_i, \mathcal{O}_{\widetilde{E}_i})^* \oplus \det \mathrm{T}_{p'_i}\mathcal{A} \otimes \mathrm{H}^0(\widetilde{E}'_i, \mathcal{O}_{\widetilde{E}'_i})^* \right\} \right)^G \\ &\simeq \left(\bigoplus_{i=5}^{10} \left\{ \det \mathrm{T}_{p_i}\mathcal{A} \oplus \det \mathrm{T}_{p'_i}\mathcal{A} \right\} \right)^G \\ &\simeq (\det V \otimes Z'')^G \simeq \det V \otimes Z \end{aligned}$$

where Z is defined via (4.6). Therefore we get an isomorphism of $\mathrm{GL}(2;\Lambda)$-modules

$$\mathrm{H}^1(X, \mathrm{T}X) \simeq (\det V \otimes \mathrm{Sym}^2\, V) \otimes Z' \oplus (\det V \otimes Z).$$

□

Corollary 4.20.

(i) Let M be a matrix in $\mathrm{GL}(2;\mathbb{Z}[\mathbf{i}])$ *such that* $M \equiv \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{pmatrix}$ *mod* $2\mathbb{Z}[\mathbf{i}]$. *Then*

$$Q_M(x) = (x + \det M)^4 \left(x + \frac{1}{\alpha^2}\right)^4 \left(x + \frac{1}{\beta^2}\right)^4 (x+1)^4.$$

(ii) There exist infinitely many M such that $\tilde{f}_M$ is rigid.

Proof.

(i) The action of $\tilde{f}_M$ on the finite set S depends only of the class of M modulo the ideal $2\mathbb{Z}[\mathbf{i}]$. Therefore, if $M \equiv \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{pmatrix}$ mod $2\mathbb{Z}[\mathbf{i}]$, the action on S'' is the permutation

$$(p_5, p'_5)\,(p_8, p'_8)\,(p_9, p_{10})(p'_9, p'_{10})$$

and the corresponding element in the group $\mathfrak{S}_6 \times (\mathbb{Z}/2\mathbb{Z})^6$ is $(9, 10), (-1, 1, 1, -1, 1, 1)$. Thus

$$\rho(M) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

so that $\mathcal{P}_M = \{1, 1, 1, -1, -1, -1\}$. On the other hand, the action on S' is trivial. Lastly, $\det M \in \{\mathbf{i}, -\mathbf{i}\}$. Hence we get the result of *(ii)*.

(ii) It suffices to prove that there are infinitely many matrices M in $\mathrm{GL}(2;\mathbb{Z}[\mathbf{i}])$ congruent to $\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{i} \end{pmatrix}$ mod $2\mathbb{Z}[\mathbf{i}]$ such that $\pm\mathbf{i}$ are not eigenvalues of M, which is straightforward: for instance the matrices $\begin{pmatrix} 2n+1 & 2\mathbf{i}n \\ -2n & -\mathbf{i}(2n-1) \end{pmatrix}$, $n \in \mathbb{Z}$ work.

□

To conclude this section, let us mention that it can be proved as in Lemma 4.3 that $-2\mathrm{K}_X$ is linearly equivalent to $\sum_{i=1}^4 E_i$, so that $|-\mathrm{K}_X| = \varnothing$ and $|-2\mathrm{K}_X| = \sum_{i=1}^4 E_i$. Hence X is a rational Coble surface [14] (that is $-2\mathrm{K}_X$ is effective but $-\mathrm{K}_X$ is not). Therefore we see that such surfaces can carry rigid automorphisms.

5. Realisation of infinitesimal deformations using divisors

Let X be a rational surface, and let f be an automorphism of X. We will present a practical method to compute the actions f^* and f_* of f on $\mathrm{H}^1(X, \mathrm{T}X)$ using divisors. This is much more delicate than for the action on the Picard group. In practical examples, the method is effective (*see* §6).

14. This fact was pointed out to us by I. Dolgachev.

5.1. **1-exceptional divisors.** Let X be a complex surface, and let D be a divisor on X. Attached to D is the holomorphic line bundle $\mathcal{O}_X(D)$ whose sections are the meromorphic fonctions f such that $D + \mathrm{div}(f)$ is effective. For any holomorphic vector bundle $\mathscr{E}$ on X, there is an exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}(D) \longrightarrow \mathscr{E}(D)_{|D} \longrightarrow 0. \tag{5.1}$$

Definition 5.1. Let X be a complex surface and let D be an effective divisor. We say that D is 1-*exceptional* if the natural morphism $\mathrm{H}^1(X, \mathrm{T}X) \longrightarrow \mathrm{H}^1(X, \mathrm{T}X(D))$ induced by the first arrow of (5.1) for $\mathscr{E} = \mathrm{T}X$ vanishes.

Remark that since $\mathrm{H}^1(\mathbb{P}^2, \mathrm{T}\mathbb{P}^2)$ vanishes, any effective divisor on $\mathbb{P}^2$ is 1-exceptional.

The importance of these divisors comes from the following immediate observation: if D is 1-exceptional, then we have an exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathrm{T}X) \longrightarrow \mathrm{H}^0(X, \mathrm{T}X(D)) \longrightarrow \mathrm{H}^0(D, \mathrm{T}X(D)_{|D}) \longrightarrow \mathrm{H}^1(X, \mathrm{T}X) \longrightarrow 0 \tag{5.2}$$

which is the long cohomology sequence associated with (5.1). Thus, if X has no nonzero holomorphic vector field,

$$\mathrm{H}^1(X, \mathrm{T}X) \simeq \frac{\mathrm{H}^0(D, \mathrm{T}X(D)_{|D})}{\mathrm{H}^0(X, \mathrm{T}X(D))}.$$

Let us now explain how to construct these divisors on rational surfaces. We fix two complex surfaces X and Y such that Y is the blowup of X at a point p. Let π be the blowup map, and let E be the exceptional divisor. There is a natural morphism of sheaves $\mathrm{T}Y \longrightarrow \pi^*\mathrm{T}X$ given by the differential of π, which is injective. By the same argument as the proof of the exactness of the sequence (4.3), the quotient sheaf $\pi^*\mathrm{T}X/\mathrm{T}Y$ is canonically isomorphic to $\iota_{E*}\mathrm{T}_E(-1)$ so that we have an exact sequence

$$0 \longrightarrow \mathrm{T}Y \longrightarrow \pi^*\mathrm{T}X \longrightarrow \iota_{E*}\mathrm{T}_E(-1) \longrightarrow 0. \tag{5.3}$$

For any locally free sheaf $\mathcal{F}$ on X, the natural pullback morphism $\mathrm{H}^i(X, \mathcal{F}) \longrightarrow \mathrm{H}^i(Y, \pi^*\mathcal{F})$ is an isomorphism; this follows by writing down the Leray spectral sequence for the sheaf $\pi^*\mathcal{F}$ and using that $\mathrm{R}^j\pi_*\mathcal{O}_Y = 0$ for $j > 0$ (*see* Lemma 4.17 *(i)* where we already used this argument). Therefore, for every divisor D on X, we have a long exact sequence

$$0 \longrightarrow \mathrm{H}^0(Y, \mathrm{T}Y(\pi^*D)) \longrightarrow \mathrm{H}^0(X, \mathrm{T}X(D)) \xrightarrow[\text{at } p]{\text{ev.}} \mathrm{T_pX} \otimes L_p \xrightarrow{\delta_D} \mathrm{H}^1(Y, \mathrm{T}Y(\pi^*D)) \longrightarrow \mathrm{H}^1(X, \mathrm{T}X(D)) \longrightarrow 0$$

where δ_D is the connection morphism, and L_p is the fiber of $\mathcal{O}_X(D)$ at p.

Proposition 5.2. *Let D be an effective divisor on X.*

(i) There is a natural isomorphism $\mathrm{H}^0(X, \mathrm{T}X(D)) \xrightarrow{\sim} \mathrm{H}^0(Y, \mathrm{T}Y(\pi^*D + E))$.

(ii) If D is 1*-exceptional on X, then $\pi^*D + E$ is* 1*-exceptional on Y.*

Proof. We have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{T}Y(\pi^*D) & \longrightarrow & \pi^*\mathrm{T}X(\pi^*D) & \longrightarrow & \iota_{E*}\mathrm{T}_E(-1) \otimes \mathcal{O}_Y(\pi^*D) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{T}Y(\pi^*D + E) & \longrightarrow & \pi^*\mathrm{T}X(\pi^*D + E) & \longrightarrow & \iota_{E*}\mathrm{T}_E(-1) \otimes \mathcal{O}_Y(\pi^*D + E) & \longrightarrow & 0 \end{array}$$

where the right vertical arrow vanishes. It follows immediately that the morphism of sheaves $\pi^*\mathrm{T}X(\pi^*D) \to \pi^*\mathrm{T}X(\pi^*D + E)$ factors through $\mathrm{T}Y(\pi^*D + E)$. This gives (i).

For (ii), let us start by proving that the composition of the connection morphism δ_D with the natural morphism $\mathrm{H}^1(Y, \mathrm{T}Y(\pi^*D)) \longrightarrow \mathrm{H}^1(Y, \mathrm{T}Y(\pi^*D + E))$ vanishes. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(\iota_{E*}\mathrm{T}_E(-1) \otimes \mathcal{O}_Y(\pi^*D)) & \xrightarrow{\delta_D} & \mathrm{H}^1(Y, \mathrm{T}Y(\pi^*D)) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(\iota_{E*}\mathrm{T}_E(-1) \otimes \mathcal{O}_Y(\pi^*D + E)) & \longrightarrow & \mathrm{H}^1(Y, \mathrm{T}Y(\pi^*D + E)) \end{array}$$

and we get our claim since the left vertical arrow vanishes. We now consider the diagram

$$\begin{array}{ccccc} & & \mathrm{H}^1(Y, \mathrm{T}Y) & \longrightarrow & \mathrm{H}^1(X, \mathrm{T}X) \\ & & \downarrow u & & \downarrow v \\ \mathrm{T}_pX \otimes L_p & \xrightarrow{\delta_D} & \mathrm{H}^1(Y, \mathrm{T}Y(\pi^*D)) & \longrightarrow & \mathrm{H}^1(X, \mathrm{T}X(D)) \end{array}$$

where the bottom line is exact. Since D is 1-exceptional, $v = 0$. This implies that the image of u lies in the image of δ_D. □

Remark 5.3. The first point of the proposition can be rephrased as follows: let Z be a section of $\mathrm{T}X(D)$. Then $Z_{|X\setminus\{p\}}$ can be considered as a section of $\mathrm{T}Y(\pi^*D)$ on $Y \setminus E$. This section extends uniquely to a section of $\mathrm{T}Y(\pi^*D + E)$.

Let us introduce some extra notations. For any effective divisor D on a surface X, we put:

$$\begin{cases} V(D) = \mathrm{H}^0(X, \mathrm{T}X(D)) \\ W(D) = \mathrm{H}^0(X, \mathrm{T}X(D)_{|D}). \end{cases}$$

We denote by $\mathfrak{h}(\mathbb{P}^2)$ the set of holomorphic vector fields on $\mathbb{P}^2$. A (possibly infinitely near) point $\widehat{P}$ in $\mathbb{P}^2$ is a point P in $\mathbb{P}^2$ together with a sequence

$$X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbb{P}^2$$

where the π_i's are point blowups and π_1 is the blowup of P. The surface X_N is called the blowup of $\mathbb{P}^2$ along $\widehat{P}$ and denoted by $\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2$. If $N = 1$ the point is simple, if $N \geq 2$ it is infinitely near. The integer N is called the length of $\widehat{P}$.

Definition 5.4. For any (possibly infinitely near) point $\widehat{P}$ in $\mathbb{P}^2$, and any effective divisor D_{base} on $\mathbb{P}^2$, we define a divisor $D_{\widehat{P}, D_{\mathrm{base}}}$ on $\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2$ as follows:

- If $(\pi_1, \dots, \pi_N)$ is the sequence of blowups defining $\widehat{P}$, let $E_1, \dots, E_N$ be the corresponding exceptional divisors on $X_1, \dots, X_N$.
- Let $D_0, \dots, D_N$ be $N + 1$ divisors on $X_0, \dots, X_N$ defined inductively by $D_0 = D_{\mathrm{base}}$ and for $1 \leq i \leq N$ $D_i = \pi_{i-1}^* D_{i-1} + E_i$.
- The divisor $D_{\widehat{P}, D_{\mathrm{base}}}$ is defined by $D_{\widehat{P}, D_{\mathrm{base}}} = D_N$.
- If D_{base} is empty we put $D_{\widehat{P}, \varnothing} = D_{\widehat{P}}$.

Note that this definition extends readily to a finite number of (possibly infinitely near) points in $\mathbb{P}^2$. According to Proposition 5.2 the divisor $D_{\widehat{P}, D_{\mathrm{base}}}$ is always 1-exceptional on $\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2$ since D_0 is 1-exceptional and $\{D_i \text{ is 1-exceptional}\} \Rightarrow \{D_{i+1} \text{ is 1-exceptional}\}$.

Lemma 5.5. *Let $\widehat{P}$ be an infinitely near point in $\mathbb{P}^2$ of length N. Then*

(i) There is a natural isomorphism $\mathfrak{h}(\mathbb{P}^2) \xrightarrow{\sim} V(D_{\widehat{P}})$.

(ii) $\dim W(D_{\widehat{P}}) = 2N$.

Proof. Using the notation of Definition 5.4, Proposition 5.2 (i) yields isomorphisms

$$\mathfrak{h}(\mathbb{P}^2) \simeq V(D_1) \simeq V(D_N) = V(D_{\widehat{P}})$$

so that using the exact sequence 5.2 we have

$$\mathrm{h}^1(X, \mathrm{T}X) = \dim W(D_{\widehat{P}}) - \Big(8 - \mathrm{h}^0(X, \mathrm{T}X)\Big).$$

Since $\mathrm{h}^1(X, \mathrm{T}X) - \mathrm{h}^0(X, \mathrm{T}X) = 2N - 8$, we get the result. □

Proposition 5.6. *Let X be a rational surface without nonzero holomorphic vector field obtained by blowing $\mathbb{P}^2$ in k (possibly infinitely near) points $\widehat{P}_1, \dots, \widehat{P}_k$. Let D_{base} be an effective divisor on $\mathbb{P}^2$, and let $V^{\dagger}(D_{\mathrm{base}})$ be a direct factor of $\mathfrak{h}(\mathbb{P}^2)$ in $V(D_{\mathrm{base}})$. We denote by $\widehat{D}$ the 1-exceptional divisor $D_{\widehat{P}_1 \cup \ldots \cup \widehat{P}_k, D_{\mathrm{base}}}$. Then there is a natural isomorphism between $V(D_{\mathrm{base}})$ and $V(\widehat{D})$, and the associated morphism*

$$\bigoplus_{i=1}^{k} W(D_{\widehat{P}_i}) \oplus V^{\dagger}(D_{\mathrm{base}}) \longrightarrow W(\widehat{D})$$

is an isomorphism.

Proof. For the first isomorphism, we argue exactly as in the proof of Lemma 5.5 (i) using Proposition 5.2 (i). For the second isomorphism, set $K = D_{\widehat{P}_1} + \ldots + D_{\widehat{P}_k}$. Let us write down the exact sequence (5.2) for the divisors K and $\widehat{D}$. Since $V(K) \simeq \mathfrak{h}(\mathbb{P}^2)$, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{h}(\mathbb{P}^2) & \longrightarrow & W(K) & \longrightarrow & \mathrm{H}^1(X, \mathrm{T}X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & V(D_{\mathrm{base}}) & \longrightarrow & W(\widehat{D}) & \longrightarrow & \mathrm{H}^1(X, \mathrm{T}X) & \longrightarrow & 0
\end{array}$$

As $V(D_{\mathrm{base}}) = \mathfrak{h}(\mathbb{P}^2) \oplus V^{\dagger}(D_{\mathrm{base}})$, we obtain that $W(\widehat{D})$ is isomorphic to $W(K) \oplus V^{\dagger}(D_{\mathrm{base}})$. □

We end the section with a statement which has no theoretical interest but which is particularly useful in practical computations:

Lemma 5.7. *Let D, D' be two divisors on a rational surface X such that:*

$$\begin{cases} D' \textit{ is } 1-\textit{exceptional.} \\ \textit{The natural map from } W(D) \text{ to } W(D') \textit{ is surjective.} \end{cases}$$

Then D is 1-exceptional.

Proof. We have a commutative diagram

$$\begin{array}{ccc} W(D) & \longrightarrow & \mathrm{H}^1(X, \mathrm{T}X) \\ \big\downarrow & & \big\| \\ W(D') & \longrightarrow & \mathrm{H}^1(X, \mathrm{T}X) \end{array}$$

where the bottom horizontal arrow is onto. Hence the top horizontal arrow is also onto. □

5.2. **Geometric bases.** In this section, we construct a basis for the vector space $W(\widehat{D})$ using coverings (the divisor $\widehat{D}$ has been introduced in Proposition 5.6), we will call such a basis a *geometric basis*. Since $\mathrm{T}X(D)_{|D}$ is the sheaf quotient $\frac{\mathrm{T}X(D)}{\mathrm{T}X}$, a global section can be represented as a family of sections $(Z_i)_{i\in I}$ of $\mathrm{T}X(D)$ on open sets U_i covering D such that $Z_i - Z_j$ is holomorphic on U_{ij} for all i, j in I.

We start with the simplest case, namely $D_{\mathrm{base}} = \varnothing$ and $k = 1$, so that $\widehat{D} = D_{\widehat{P}}$. Let us construct this basis by induction on the length N of $\widehat{P}$.

Let $\widehat{P}$ be an infinitely near point of $\mathbb{P}^2$ of length N, let q be a point in one of the exceptional divisors, let $\widehat{P}' = \widehat{P} \cup \{q\}$, and put $X = \mathrm{Bl}_{\widehat{P}}\mathbb{P}^2$ and $Y = \mathrm{Bl}_{\widehat{P}'}\mathbb{P}^2$. Assume that we are given a covering of $D_{\widehat{P}}$ by open sets $(U_i)_{i\in I}$ of X such that :

① there exists a unique i_0 in I such that $q \in U_{i_0}$;

② the evaluation map $\mathrm{H}^0(U_{i_0}, \mathrm{T}U_{i_0}) \longrightarrow \mathrm{T}_qX$ is surjective;

③ there exists a basis $Z_1, \ldots, Z_{2N}$ of $W(D_{\widehat{P}})$, where for each α, Z_α is a collection of holomorphic sections $(Z_{i\alpha})_{i\in I}$ of $\mathrm{T}X(D_{\widehat{P}})$ on the U_i's such that for all i and j, the section $Z_{i\alpha} - Z_{j\alpha}$ is holomorphic on each $U_{i\alpha} \cap U_{j\alpha}$.

Let $\pi\colon Y \to X$ be the blowup map, and let E be the exceptional divisor. Set $U'_i = \pi^{-1}(U_i)$. For any basis (v_1, v_2) of $\mathrm{T}_q(\mathrm{Bl}_{\widehat{P}}\mathbb{P}^2)$, thanks to ②, we can choose two holomorphic vector fields T_1, T_2 on U_{i_0} which extend (v_1, v_2). We will consider the vector fields T_1 and T_2 as sections of $\mathrm{T}Y(E)$ on U'_{i_0}. Let us now consider the sections $(Z'_1, \ldots, Z'_{2N+2})$ of $W(D_{\widehat{P}'})$ defined using the covering $(U'_i)_{i\in I}$ as follows :

	Z'_1	……	Z'_{2N}	Z'_{2N+1}	Z'_{2N+2}
U'_{i_0}	Z_{1i_0}	……	Z_{2Ni_0}	T_1	T_2
$U'_j \quad j \neq i_0$	Z_{1j}	……	Z_{2Nj}	0	0

Then we have the following result:

Proposition 5.8. *The family $(Z'_1, \ldots, Z'_{2N+2})$ is a basis of $W(D_{\widehat{P}'})$.*

Proof. Thanks to Lemma 5.5, it suffices to show that this family is free. Let $\lambda_1, \dots, \lambda_{2N+2}$ be complex numbers such that $\lambda_1 Z'_1 + \lambda_2 Z'_2 + \dots + \lambda_{2N+2} Z'_{2N+2} = 0$. Then for all $j \neq i_0$,

$$\lambda_1 Z'_{1j} + \lambda_2 Z'_{2j} + \dots + \lambda_{2N} Z'_{2Nj}$$

is holomorphic on U'_j, and $\lambda_1 Z'_{1i_0} + \lambda_2 Z'_{2i_0} + \dots + \lambda_{2N} Z'_{2Ni_0} + \lambda_{2N+1} T_1 + \lambda_{2N+2} T_2$ is holomorphic on U'_{i_0}.

Therefore, $\lambda_1 Z'_{1j} + \lambda_2 Z'_{2j} + \dots + \lambda_{2N} Z'_{2Nj}$ is holomorphic on U'_j also for $j = i_0$ and $\lambda_{2N+1} T_1 + \lambda_{2N+2} T_2$ (considered as a vector field on U_{i_0}) vanishes at q. Since $(Z_1, \dots, Z_{2N})$ is a basis of $W(D_{\widehat{P}})$ we get $\lambda_1 = \dots = \lambda_{2N} = 0$; and as (v_1, v_2) is a basis of $\mathrm{T}_q X$, $\lambda_{2N+1} = \lambda_{2N+2} = 0$. □

Remarks 5.9.

(i) In practical computations, U_{i_0} is a domain for some holomorphic coordinates (z, w), $T_1 = \partial_z$ and $T_2 = \partial_w$.

(ii) If q' is a point on $\widehat{P'}$, the covering $(U'_i)_{i\in I}$ does not satisfy in general conditions ② and ③. Therefore it is necessary to refine the covering at each blowup.

Using Proposition 5.6, we can now easily construct a basis of $W(\widehat{D})$. Indeed, we have already explained how to construct a basis of each $W(D_{\widehat{P}_k})$. Then it suffices to take a basis of $V(D_{\mathrm{base}})^{\dagger}$ and to pull back these meromorphic vector fields to the surface X.

5.3. **Algebraic bases.** In practical computations, if an element of $W(\widehat{D})$ is given by local meromorphic vector fields on open sets of a covering, it is not a priori obvious to decompose this element in a geometric basis (as constructed in §5.2). In this section, we will construct another basis (we call it an algebraic basis) which solves this problem.

We start by defining the residue morphisms. Let $\widehat{P} = \{p_1, \dots, p_N\}$, and let $\overline{E}_1, \dots, \overline{E}_N$ be the corresponding strict transforms of the exceptional divisors. We fix holomorphic coordinates (x_i, y_i) near p_i such that $p_i = (0, 0)$ in these coordinates, and we introduce new holomorphic coordinates (u_i, v_i) by putting $x_i = u_i$ and $y_i = u_i v_i$. We can consider (u_i, v_i) as holomorphic coordinates on an open subset of X which contains $\overline{E}_i$ except a finite number of points.

Let $(0, \lambda_i)$ be a generic point of $\overline{E}_i$, let Z be an element of $W(D_{\widehat{P}})$, and let Z_i be a section of $\mathrm{T}X(D)$ which lifts Z near this point. Then Z_i admits a Laurent expansion

$$Z_i(u_i, v_i) = \sum_{n=1}^{n_0} \sum_{m=0}^{\infty} \frac{(v_i - \lambda_i)^m (a_{nm}\, \partial_u + b_{nm}\, \partial_v)}{u^n} + \{\text{holomorphic terms}\}.$$

Definition 5.10. For any generic complex number λ_i, we define the i^{th} *residue morphism*

$$\mathrm{res}_{E_i} \colon W(D_{\widehat{P}}) \longrightarrow \mathbb{C}^2$$

by the formula $\mathrm{res}_{E_i}(Z) = (b_{10}, b_{11})$. We also define $\mathrm{res}_{\widehat{P}} \colon W(D_{\widehat{P}}) \longrightarrow \mathbb{C}^{2N}$ by $\mathrm{res}_{\widehat{P}} = \bigoplus_{i=1}^{N} \mathrm{res}_{E_i}$.

Remark 5.11. The definition of these residues depends on the coordinates and on the generic parameters on the exceptional divisors.

Lemma 5.12. *The morphism* $\mathrm{res}_{\widehat{P}}$ *is an isomorphism.*

Proof. Let us take a geometric basis associated to the coordinates $(x_i, y_i)_{1\le i\le N}$ as constructed in §5.2. Then the matrix of $\mathrm{res}_{\widehat{P}}$ is lower triangular by blocks, where the diagonal blocks are the 2×2 matrices whose columns are the vectors $\mathrm{res}_{E_i}(\partial_{x_i})$ and $\mathrm{res}_{E_i}(\partial_{y_i})$. We have $\partial_{x_i} = \partial_{u_i} - \frac{v_i}{u_i}\partial_{v_i}$ and $\partial_{y_i} = \frac{1}{u_i}\partial_{v_i}$. Therefore we obtain $\mathrm{res}_{E_i}(\partial_{x_i}) = (-1, -\lambda_i)$ and $\mathrm{res}_{E_i}(\partial_{y_i}) = (1, 0)$, so that the diagonal blocks are all invertible. □

We can now define algebraic basis in the general case. We take the notation of Proposition 5.6. In order to avoid cumbersome notation, we will assume for simplicity that D_{base} is irreducible, and leave the general case to the reader.

Let $\chi_1, \ldots, \chi_r$ be a basis of $V(D_{\mathrm{base}})^{\dagger}$.

Lemma 5.13. *Assume that D_{base} is irreducible, and let Z be an element of $W(\widehat{D})$. Then there exist unique complex numbers $\alpha_1(Z), \ldots, \alpha_r(Z)$ such that for any generic point ξ of D_{base} and any lift of $\widetilde{Z}$ of Z near ξ (with respect to the sheaf morphism $\mathrm{T}X(D) \to \mathrm{T}X(D)_{|D}$),*

$$\widetilde{Z} = \alpha_1(Z)\, \chi_1 + \ldots + \alpha_m(Z)\, \chi_r + \text{ holomorphic terms}$$

in a neighborhood of ξ .

Proof. The existence of such a decomposition is straightforward: the $\alpha_i(Z)$'s are the coefficients of Z in $V(D_{\mathrm{base}})^{\dagger}$ when decomposed in a geometric basis. For the unicity, we remark that $\alpha_1\, \chi_1 + \ldots + \alpha_r\, \chi_r$ is holomorphic near a point of D_{base}, it must be holomorphic on $\mathbb{P}^2$ since D_{base} is irreducible. Since $V(D_{\mathrm{base}})^{\dagger}$ is a direct factor of $\mathfrak{h}(\mathbb{P}^2)$, all the coefficients α_i must vanish. □

Corollary 5.14. *With the notations of Proposition 5.6, the map*

$$(\mathrm{res}_{\widehat{P}_1}, \ldots, \mathrm{res}_{\widehat{P}_k}, \alpha_1, \ldots, \alpha_r)\colon\, W(\widehat{D}) \longrightarrow \mathbb{C}^{2N_1+\ldots+2N_k+r}$$

is an isomorphism.

By definition, the associated algebraic basis of $W(\widehat{D})$ is the image of the canonical basis of $\mathbb{C}^{2N_1+\ldots+2N_k+r}$ by the inverse isomorphism.

6. An explicit example on $\mathbb{P}^2$ blown up in 15 points

6.1. **The strategy.** Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the birational map given by

$$(x : y : z) \to (xz^2 + y^3 : yz^2 : z^3).$$

The map ϕ induces an isomorphism (which we still denote by ϕ) between $\mathrm{Bl}_{\widehat{P}_1}\,\mathbb{P}^2$ and $\mathrm{Bl}_{\widehat{P}_2}\,\mathbb{P}^2$, where $\widehat{P}_1$ and $\widehat{P}_2$ are infinitely near points of $\mathbb{P}^2$ of length 5 centered at $(1 : 0 : 0)$ described in [13, §2.1]. For any complex number α, we put

$$A = \begin{pmatrix} \alpha & 2(1-\alpha) & 2+\alpha-\alpha^2 \\ -1 & 0 & \alpha+1 \\ 1 & -2 & 1-\alpha \end{pmatrix}$$

and we consider it as an element of $\mathrm{PGL}(3;\mathbb{C})$. Then the map $A\phi$ lifts to an automorphism ψ of the rational surface X obtained by blowing up the projective plane 15 times at $\widehat{P}_1$, $A\widehat{P}_2$ and $A\phi A\widehat{P}_2$

(*see* [13, §3.3]). Recall that the parameter α is not really interesting because two different values of α correspond to linearly conjugate automorphisms, however we keep it in order to check some calculations in the sequel. Our aim is to compute the map ψ_* acting on $\mathrm{H}^1(X, \mathrm{T}X)$.

Let us recall the construction of $\widehat{P}_1$ (resp. $\widehat{P}_2$). In affine coordinates (y, z), we blow up the point $(0, 0)_{y,z}$ in $\mathbb{P}^2$ and we denote by E the exceptional divisor. We put $y = u_1$, $z = u_1 v_1$, we blow up $(0, 0)_{u_1,v_1}$ and F is the exceptional divisor. Then we put $u_1 = r_2 s_2$, $v_1 = s_2$, we blow up $(0, 0)_{r_2,s_2}$ and we denote by G the exceptional divisor. Next, we put $r_2 = r_3 s_3$, $s_2 = s_3$, we blow up $(-1, 0)_{r_3,s_3}$ (resp. $(1, 0)_{r_3,s_3}$) and H (resp. K) is the exceptional divisor. Then, we put $r_3 = r_4 s_4 - 1$, $s_3 = s_4$ (resp. $r_3 = c_4 d_4 + 1$, $s_3 = d_4$), we blow up $(0, 0)_{r_4,s_4}$ (resp. $(0, 0)_{c_4,d_4}$) and we denote by L (resp. M) the last exceptional divisor. Lastly, we put $r_4 = r_5 s_5$, $s_4 = s_5$ (resp. $c_4 = c_5 d_5$, $d_4 = d_5$).

Set $X_1 = \mathrm{Bl}_{\widehat{P}_1} \mathbb{P}^2$ and $X_2 = \mathrm{Bl}_{\widehat{P}_2} \mathbb{P}^2$. Let $\overline{\Delta}$ be the strict transform of the line $\Delta = \{z = 0\}$ in X_2. Then

$$\phi(E) = E, \quad \phi(F) = K, \quad \phi(G) = G, \quad \phi(H) = F, \quad \phi(L) = \overline{\Delta}\,. \tag{6.1}$$

We have $D_{\widehat{P}_1} = E + 2F + 4G + 5H + 6L$ and $D_{\widehat{P}_2} = E + 2F + 4G + 5K + 6M$. Let D_1 be the divisor on X_1 given by

$$D_1 = E + 2F + 3G + 4H + 5L.$$

By explicit calculation, the natural morphism from $W(D_1)$ to $W(D_{\widehat{P}_1})$ is surjective so that D_1 is 1-exceptional on X_1 by Lemma 5.7. We now define a divisor D_2 on X_2 as follows (*see* Definition 5.4):

$$D_2 = D_{\widehat{P}_2, 5\overline{\Delta}}.$$

By (6.1), we have $\phi_* D_1 \leq D_2$. Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be the two 1-exceptional divisors on X given by:

$$\mathfrak{D}_1 = D_1 + AD_{\widehat{P}_2} + A\phi A\, D_{\widehat{P}_2}, \qquad \mathfrak{D}_2 = D_{\widehat{P}_1} + AD_2 + A\phi A\, D_{\widehat{P}_2}.$$

Then $\mathfrak{D}_1 \leq \mathfrak{D}_2$ and $f_* \mathfrak{D}_1 \leq \mathfrak{D}_2$. Therefore the morphism ψ_* acting on $\mathrm{H}^1(X, \mathrm{T}X)$ can be obtained as the composition

$$\frac{W(\mathfrak{D}_1)}{V(\mathfrak{D}_1)} \xrightarrow{\psi_*} \frac{W(\mathfrak{D}_2)}{V(\mathfrak{D}_2)} \xrightarrow{\sim} \frac{W(\mathfrak{D}_1)}{V(\mathfrak{D}_1)} \tag{6.2}$$

where the inverse of last arrow is induced by the natural morphism from $W(\mathfrak{D}_1)$ to $W(\mathfrak{D}_2)$. To compute ψ_*, the strategy runs as follows:

Calculations for the pair (D_1, D_2).

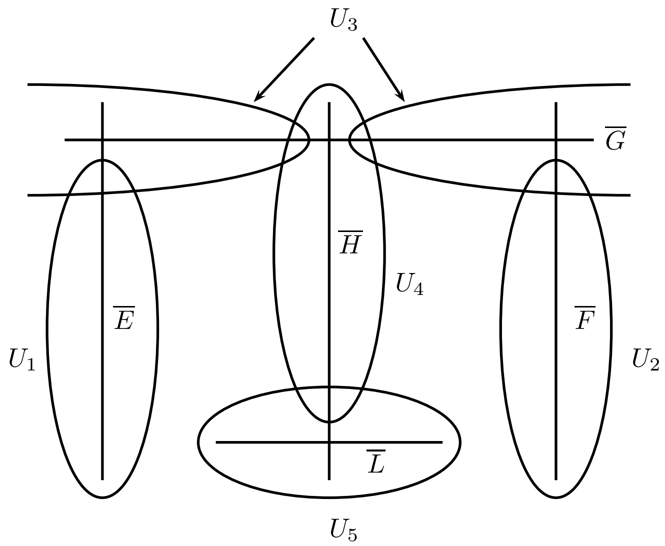
- *Step 1* – Express the vectors of a geometric basis of $W(D_1)$ in an algebraic basis.
- *Step 2* – Compute $\phi_*\colon W(D_1) \longrightarrow W(D_2)$ where $W(D_1)$ is endowed with a geometric basis and $W(D_2)$ is endowed with an algebraic basis.
- *Step 3* – Find bases of $V(D_1)$ and $V(D_2)$ whose vectors are expressed in algebraic bases of $W(D_1)$ and $W(D_2)$ respectively.

Calculations for the pair $(\mathfrak{D}_1, \mathfrak{D}_2)$.

- *Step 1* – Compute $\psi_*\colon W(\mathfrak{D}_1) \longrightarrow W(\mathfrak{D}_2)$ where $W(\mathfrak{D}_1)$ and $W(\mathfrak{D}_2)$ are endowed with algebraic bases.
- *Step 2* – Find bases of $V(\mathfrak{D}_1)$ and $V(\mathfrak{D}_2)$ whose vectors are expressed in algebraic bases of $W(\mathfrak{D}_1)$ and $W(\mathfrak{D}_2)$ respectively.

6.2. **Calculations for the pair** (D_1, D_2)**.** We give results of the calculations for the first three steps listed above.

Step 1 – We use the coordinates (y, z), (u_1, v_1), (r_2, s_2), (r_3, s_3), (r_4, s_4) and (r_5, s_5) to compute the algebraic and geometric bases on X_1. Let $(U_i)_{1\le i\le 5}$ be the covering of D_1 given by the following picture:



Then a geometric basis of $W(D_1)$ is given by the ten following vectors:

	U_1	U_2	U_3	U_4	U_5
$\mathfrak{e}_1$	∂y	∂y	∂y	∂y	∂y
$\mathfrak{e}_2$	∂z	∂z	∂z	∂z	∂z
$\mathfrak{e}_3$	0	∂u_1	∂u_1	∂u_1	∂u_1
$\mathfrak{e}_4$	0	∂v_1	∂v_1	∂v_1	∂v_1
$\mathfrak{e}_5$	0	0	∂r_2	∂r_2	∂r_2
$\mathfrak{e}_6$	0	0	∂s_2	∂s_2	∂s_2
$\mathfrak{e}_7$	0	0	0	∂r_3	∂r_3
$\mathfrak{e}_8$	0	0	0	∂s_3	∂s_3
$\mathfrak{e}_9$	0	0	0	0	∂r_4
$\mathfrak{e}_{10}$	0	0	0	0	∂s_4

– Matrix $\mathscr{P}$ –

If λ_1, λ_2, λ_3, λ_4 and λ_5 are generic parameters on E, F, G, H and L respectively and if $(\mathfrak{l}_i)_{1\le i\le 10}$ is the associated algebraic basis, a direct calculation yields:

	$\mathfrak{e}_1$	$\mathfrak{e}_2$	$\mathfrak{e}_3$	$\mathfrak{e}_4$	$\mathfrak{e}_5$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$
$\mathfrak{l}_1$	$-\lambda_1$	1	0	0	0	0	0	0	0	0
$\mathfrak{l}_2$	-1	0	0	0	0	0	0	0	0	0
$\mathfrak{l}_3$	2	0	1	$-\lambda_2$	0	0	0	0	0	0
$\mathfrak{l}_4$	0	0	0	-1	0	0	0	0	0	0
$\mathfrak{l}_5$	0	0	0	$-2\lambda_3$	1	$-\lambda_3$	0	0	0	0
$\mathfrak{l}_6$	0	0	0	-2	0	-1	0	0	0	0
$\mathfrak{l}_7$	$-\lambda_4^2$	λ_4^3	0	$-3\lambda_4$	0	$-2\lambda_4$	1	$-\lambda_4$	0	0
$\mathfrak{l}_8$	$-2\lambda_4$	$3\lambda_4^2$	0	-3	0	-2	0	-1	0	0
$\mathfrak{l}_9$	0	$2\lambda_5^2$	0	$-4\lambda_5$	0	$-3\lambda_5$	0	$-2\lambda_5$	1	$-\lambda_5$
$\mathfrak{l}_{10}$	0	$4\lambda_5$	0	-4	0	-3	0	-2	0	-1

– *Matrix* $\mathscr{K}$ –

Step 2 – We start by fixing a basis of $\mathrm{H}^0(\mathbb{P}^2, \mathrm{T}\mathbb{P}^2(m\Delta))$. We divide $\mathrm{H}^0(\mathbb{P}^2, \mathrm{T}\mathbb{P}^2(m\Delta))$ in four subspaces, the first one corresponding to holomorphic vector fields.

–Vector fields of type A that span a subspace of dimension 8:

$$a_1 = \partial_y, \quad a_2 = y\,\partial_y, \quad a_3 = z\,\partial_y, \quad a_4 = \partial_z, \quad a_5 = y\,\partial_z, \quad a_6 = z\,\partial_z, \quad a_7 = yz\,\partial_z, \quad a_8 = z^2\,\partial_z.$$

–Vector fields of type B that span a subspace of dimension $\frac{m(m+5)}{2}$:

$$b_{p,q} = \frac{y^p}{z^q}\,\partial_y \qquad 1 \leq q \leq m, \quad 0 \leq p \leq q+1.$$

–Vector fields of type C that span a subspace of dimension $\frac{m(m+5)}{2}$:

$$c_{p,q} = \frac{y^p}{z^q}\,\partial_z \qquad 1 \leq q \leq m, \quad 0 \leq p \leq q+1.$$

–Vector fields of type D that span a subspace of dimension m:

$$d_p = \frac{y^{p+2}}{z^p}\,\partial_y + \frac{y^{p+1}}{z^{p-1}}\,\partial_z \qquad 1 \leq p \leq m.$$

We take for $\mathrm{H}^0(\mathbb{P}^2, \mathrm{T}\mathbb{P}^2(m\Delta))^{\dagger}$ the subspace spanned by meromorphic vector fields of type B, C, and D (in our example, $m = 5$). Then we use the coordinates (y, z), (u_1, v_1), (r_2, s_2), (r_3, s_3), (c_4, d_4) and (c_5, d_5) on X_2. If μ_1, μ_2, μ_3, μ_4 and μ_5 are the generic parameters on E, F, G, K and M respectively, we consider an algebraic basis of $W(D_2)$ associated with these parameters consisting of ten vectors $(\mathfrak{m}_i)_{1 \leq i \leq 10}$ corresponding to the exceptional divisors, and 55 vectors (25 of type B, 25 of type C and 5 of type D) corresponding to a basis of $\mathrm{H}^0(\mathbb{P}^2, \mathrm{T}\mathbb{P}^2(5\Delta))^{\dagger}$.

By direct computation, we have

$$\phi_*\partial_y = \left(1 - \frac{4y^3}{z^2} + \frac{3y^6}{z^4}\right)\partial_y + \left(-\frac{3y^2}{z} + \frac{3y^5}{z^3}\right)\partial_z, \qquad \phi_*\partial_z = 2\left(\frac{y^4}{z^3} - \frac{y^7}{z^5}\right)\partial_y + \left(1 + \frac{y^3}{z^2} - \frac{2y^6}{z^4}\right)\partial_z,$$

$$\phi_*\partial_{u_1} = \left(1 - 2\frac{y^3}{z^2} + \frac{y^6}{z^4}\right)\partial_y + \left(\frac{z}{y} - 2\frac{y^2}{z} + \frac{y^5}{z^3}\right)\partial_z, \qquad \phi_*\partial_{v_1} = 2\frac{y^5}{z^3}\,\partial_y + \left(y + 2\frac{y^4}{z^2}\right)\partial_z,$$

$$\phi_*\partial_{r_2} = \left(\frac{z}{y} - 2\frac{y^2}{z} + \frac{y^5}{z^3}\right)\partial_y + \left(\frac{z^2}{y^2} - 2y + \frac{y^4}{z^2}\right)\partial_z, \qquad \phi_*\partial_{s_2} = \left(\frac{y^2}{z} + \frac{y^5}{z^3}\right)\partial_y + \left(2y + \frac{y^4}{z^2}\right)\partial_z,$$

$$\phi_*\partial_{r_3} = \left(\frac{z^2}{y^2} - 2y + \frac{y^4}{z^2}\right)\partial_y + \left(\frac{z^3}{y^3} - 2z + \frac{y^3}{z}\right)\partial_z, \qquad \phi_*\partial_{s_3} = \frac{2y^2}{z}\,\partial_y + 3y\,\partial_z,$$

$$\phi_*\partial_{r_4} = \left(\frac{z^3}{y^3} - 2z + \frac{y^3}{z}\right)\partial_y + \left(\frac{z^4}{y^4} - \frac{2z^2}{y} + y^2\right)\partial_z, \qquad \phi_*\partial_{s_4} = \left(\frac{z}{y} + \frac{y^2}{z}\right)\partial_y + \left(\frac{z^2}{y^2} + 2y\right)\partial_z.$$

Therefore, the matrix of $\phi_*\colon W(D_1) \longrightarrow W(D_2)$ is:

	$\phi_*\mathfrak{e}_1$	$\phi_*\mathfrak{e}_2$	$\phi_*\mathfrak{e}_3$	$\phi_*\mathfrak{e}_4$	$\phi_*\mathfrak{e}_5$	$\phi_*\mathfrak{e}_6$	$\phi_*\mathfrak{e}_7$	$\phi_*\mathfrak{e}_8$	$\phi_*\mathfrak{e}_9$	$\phi_*\mathfrak{e}_{10}$
$\mathfrak{m}_1$	$-\mu_1$	1	0	0	0	0	0	0	0	0
$\mathfrak{m}_2$	-1	0	0	0	0	0		0	0	0
$\mathfrak{m}_3$	2	0	1	$-\mu_2$	$-2\mu_2$	0	μ_2^2	μ_2	0	0
$\mathfrak{m}_4$	0	0	0	-1	-2	0	$2\mu_2$	1	0	0
$\mathfrak{m}_5$	0	0	0	$2\mu_3(\mu_3-1)$	$(\mu_3-1)^2$	$\mu_3(\mu_3-1)$	0	0	0	0
$\mathfrak{m}_6$	0	0	0	$4\mu_3-2$	$2(\mu_3-1)$	$2\mu_3-1$	0	0	0	0
$\mathfrak{m}_7$	$2\mu_4^2$	$-\mu_4^3$	μ_4^2	μ_4	0	0	0	0	0	0
$\mathfrak{m}_8$	$4\mu_4$	$-3\mu_4^2$	$2\mu_4$	1	0	0	0	0	0	0
$\mathfrak{m}_9$	0	0	0	0	0	0	0	0	0	0
$\mathfrak{m}_{10}$	0	0	0	0	0	0	0	0	0	0
$\mathrm{H}^0\big(\mathbb{P}^2, \mathrm{T}\mathbb{P}^2(5\Delta)\big)^\dagger$	$-4b_{3,2}$ $-3c_{2,1}$ $3d_4$	$2b_{4,3}$ $c_{3,2}$ $-2d_5$	$-2b_{3,2}$ $-2c_{2,1}$ d_4	$2d_3$	$-2b_{2,1}$ d_3	$b_{2,1}$ d_3	d_2	$2b_{2,1}$	d_1	$b_{2,1}$

– Matrix $\mathscr{L}$ –

Step 3 – For typographical reasons, we will write the vectors of $V(D_1)$ and $V(D_2)$ as lines (and not as columns).

	$\mathfrak{l}_1$	$\mathfrak{l}_2$	$\mathfrak{l}_3$	$\mathfrak{l}_4$	$\mathfrak{l}_5$	$\mathfrak{l}_6$	$\mathfrak{l}_7$	$\mathfrak{l}_8$	$\mathfrak{l}_9$	$\mathfrak{l}_{10}$
a_1	$-\lambda_1$	-1	2	0	0	0	$-\lambda_4^2$	$-2\lambda_4$	0	0
a_2	0	0	0	0	0	0	-3	0	0	0
a_3	0	0	0	0	0	0	0	0	-3	0
a_4	1	0	0	0	0	0	λ_4^3	$3\lambda_4^2$	$2\lambda_5^2$	$4\lambda_5$
a_5	0	0	$-\lambda_2$	-1	$-2\lambda_3$	-2	$-3\lambda_4$	-3	$-4\lambda_5$	-4
a_6	0	0	0	0	0	0	2	0	0	0
a_7	0	0	0	0	0	0	0	0	0	0
a_8	0	0	0	0	0	0	0	0	0	0

– *Matrix* ${}^t\mathscr{M}$ –

	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$	$\mathfrak{m}_4$	$\mathfrak{m}_5$	$\mathfrak{m}_6$	$\mathfrak{m}_7$	$\mathfrak{m}_8$	$\mathfrak{m}_9$	$\mathfrak{m}_{10}$
a_1	$-\mu_1$	-1	2	0	0	0	$-\mu_4^2$	$-2\mu_4$	0	0
a_2	0	0	0	0	0	0	3	0	0	0
a_3	0	0	0	0	0	0	0	0	3	0
a_4	1	0	0	0	0	0	$-\mu_4^3$	$-3\mu_4^2$	$2\mu_5^2$	$4\mu_5$
a_5	0	0	$-\mu_2$	-1	$-2\mu_3$	-2	$-3\mu_4$	-3	$-4\mu_5$	-4
a_6	0	0	0	0	0	0	-2	0	0	0
a_7	0	0	0	0	0	0	0	0	0	0
a_8	0	0	0	0	0	0	0	0	0	0

– *Matrix* ${}^t\mathscr{N}_{\mathrm{a}}$ –

	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$	$\mathfrak{m}_4$	$\mathfrak{m}_5$	$\mathfrak{m}_6$	$\mathfrak{m}_7$	$\mathfrak{m}_8$	$\mathfrak{m}_9$	$\mathfrak{m}_{10}$
$b_{0,1}$	0	0	0	0	0	0	$2\mu_4^5$	$10\mu_4^4$	$3\mu_5^3$	$9\mu_5^2$
$b_{1,1}$	-1	0	0	0	0	0	μ_4^3	$3\mu_4^2$	$-2\mu_5^2$	$-4\mu_5$
$b_{2,1}$	0	0	$2\mu_2$	2	$3\mu_3$	3	$4\mu_4$	4	$5\mu_5$	5
$b_{0,2}$	0	0	0	0	0	0	$-9\mu_4^8$	$-72\mu_4^7$	0	0
$b_{1,2}$	0	0	0	0	0	0	$-3\mu_4^6$	$-18\mu_4^5$	0	0
$b_{2,2}$	$-\mu_1{}^{-1}$	μ_1^{-2}	0	0	0	0	$-\mu_4^4$	$-4\mu_4^3$	0	0
$b_{3,2}$	0	0	0	0	0	0	0	0	0	0
$b_{0,3}$	0	0	0	0	0	0	$52\mu_4^{11}$	$572\mu_4^{10}$	$-28\mu_5^6$	$-168\mu_5^5$
$b_{1,3}$	0	0	0	0	0	0	$15\mu_4^9$	$135\mu_4^8$	$12\mu_5^5$	$60\mu_5^4$
$b_{2,3}$	0	0	0	0	0	0	$4\mu_4^7$	$28\mu_4^6$	$-5\mu_5^4$	$-20\mu_5^3$
$b_{3,3}$	$-\mu_1^{-2}$	$2\mu_1^{-3}$	0	0	0	0	μ_4^5	$5\mu_4^4$	$2\mu_5^3$	$6\mu_5^2$
$b_{4,3}$	0	0	0	0	0	0	0	0	0	0
$b_{0,4}$	0	0	0	0	0	0	$-340\mu_4^{14}$	$-4760\mu_4^{13}$	0	0
$b_{1,4}$	0	0	0	0	0	0	$-91\mu_4^{12}$	$-1092\mu_4^{11}$	0	0
$b_{2,4}$	0	0	0	0	0	0	$-22\mu_4^{10}$	$-220\mu_4^9$	0	0
$b_{3,4}$	0	0	0	0	0	0	$-5\mu_4^8$	$-40\mu_4^7$	0	0
$b_{4,4}$	$-\mu_1^{-3}$	$3\mu_1^{-4}$	0	0	0	0	$-\mu_4^6$	$-6\mu_4^5$	0	0
$b_{5,4}$	0	0	0	0	0	0	0	0	0	0
$b_{0,5}$	0	0	0	0	0	0	$2394\mu_4^{17}$	$40698\mu_4^{16}$	$429\mu_5^9$	$3861\mu_5^8$
$b_{1,5}$	0	0	0	0	0	0	$612\mu_4^{15}$	$9180\mu_4^{14}$	$-165\mu_5^8$	$-1320\mu_5^7$
$b_{2,5}$	0	0	0	0	0	0	$140\mu_4^{13}$	$1820\mu_4^{12}$	$60\mu_5^7$	$420\mu_5^6$
$b_{3,5}$	0	0	0	0	0	0	$30\mu_4^{11}$	$330\mu_4^{10}$	$-21\mu_5^6$	$-126\mu_5^5$
$b_{4,5}$	0	0	0	0	0	0	$6\mu_4^9$	$54\mu_4^8$	$7\mu_5^5$	$35\mu_5^4$
$b_{5,5}$	$-\mu_1^{-4}$	$4\mu_1^{-5}$	0	0	0	0	μ_4^7	$7\mu_4^6$	$-2\mu_5^4$	$-8\mu_5^3$
$b_{6,5}$	0	0	0	0	0	0	0	0	0	0

– *Matrix* ${}^t\mathscr{N}_{\mathrm{b}}$ –

	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$	$\mathfrak{m}_4$	$\mathfrak{m}_5$	$\mathfrak{m}_6$	$\mathfrak{m}_7$	$\mathfrak{m}_8$	$\mathfrak{m}_9$	$\mathfrak{m}_{10}$
$c_{0,1}$	0	0	0	0	0	0	$4\mu_4^6$	$24\mu_4^5$	0	0
$c_{1,1}$	μ_1^{-1}	$-\mu_1^{-2}$	0	0	0	0	μ_4^4	$4\mu_4^3$	0	0
$c_{2,1}$	0	0	0	0	0	0	0	0	0	0
$c_{0,2}$	0	0	0	0	0	0	$-25\mu_4^9$	$-225\mu_4^8$	$-18\mu_5^5$	$-90\mu_5^4$
$c_{1,2}$	0	0	0	0	0	0	$-5\mu_4^7$	$-35\mu_4^6$	$6\mu_5^4$	$24\mu_5^3$
$c_{2,2}$	μ_1^{-2}	$-2\mu_1^{-3}$	0	0	0	0	$-\mu_4^5$	$-5\mu_4^4$	$-2\mu_5^3$	$-6\mu_5^2$
$c_{3,2}$	0	0	0	0	0	0	0	0	0	0
$c_{0,3}$	0	0	0	0	0	0	$182\mu_4^{12}$	$2184\mu_4^{11}$	0	0
$c_{1,3}$	0	0	0	0	0	0	$33\mu_4^{10}$	$330\mu_4^9$	0	0
$c_{2,3}$	0	0	0	0	0	0	$6\mu_4^8$	$48\mu_4^7$	0	0
$c_{3,3}$	μ_1^{-3}	$-3\mu_1^{-4}$	0	0	0	0	μ_4^6	$6\mu_4^5$	0	0
$c_{4,3}$	0	0	0	0	0	0	0	0	0	0
$c_{0,4}$	0	0	0	0	0	0	$-1428\mu_4^{15}$	$-21420\mu_4^{14}$	$330\mu_5^8$	$2640\mu_5^7$
$c_{1,4}$	0	0	0	0	0	0	$-245\mu_4^{13}$	$-3185\mu_4^{12}$	$-96\mu_5^7$	$-672\mu_5^6$
$c_{2,4}$	0	0	0	0	0	0	$-42\mu_4^{11}$	$-462\mu_4^{10}$	$28\mu_5^6$	$168\mu_5^5$
$c_{3,4}$	0	0	0	0	0	0	$-7\mu_4^9$	$-63\mu_4^8$	$-8\mu_5^5$	$-40\mu_5^4$
$c_{4,4}$	μ_1^{-4}	$-4\mu_1^{-5}$	0	0	0	0	$-\mu_4^7$	$-7\mu_4^6$	$2\mu_5^4$	$8\mu_5^3$
$c_{5,4}$	0	0	0	0	0	0	0	0	0	0
$c_{0,5}$	0	0	0	0	0	0	$11704\mu_4^{18}$	$210672\mu_4^{17}$	0	0
$c_{1,5}$	0	0	0	0	0	0	$1938\mu_4^{16}$	$31008\mu_4^{15}$	0	0
$c_{2,5}$	0	0	0	0	0	0	$320\mu_4^{14}$	$4480\mu_4^{13}$	0	0
$c_{3,5}$	0	0	0	0	0	0	$52\mu_4^{12}$	$624\mu_4^{11}$	0	0
$c_{4,5}$	0	0	0	0	0	0	$8\mu_4^{10}$	$80\mu_4^9$	0	0
$c_{5,5}$	μ_1^{-5}	$-5\mu_1^{-6}$	0	0	0	0	μ_4^8	$8\mu_4^7$	0	0
$c_{6,5}$	0	0	0	0	0	0	0	0	0	0

– *Matrix* ${}^t\mathscr{N}_{\mathrm{c}}$ –

	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$	$\mathfrak{m}_4$	$\mathfrak{m}_5$	$\mathfrak{m}_6$	$\mathfrak{m}_7$	$\mathfrak{m}_8$	$\mathfrak{m}_9$	$\mathfrak{m}_{10}$
d_1	0	0	0	0	0	0	0	0	1	0
d_2	0	0	μ_2^2	$2\mu_2$	0	0	1	0	0	0
d_3	0	0	0	0	μ_3^2	$2\mu_3$	$2\mu_4$	2	$2\mu_5$	2
d_4	0	0	0	0	0	0	μ_4^2	$2\mu_4$	0	0
d_5	0	0	0	0	0	0	0	0	μ_5^2	$2\mu_5$

– *Matrix* ${}^t\mathscr{N}_{\mathrm{d}}$ –

Although we won't need it, it is easy to verify that

$$\begin{cases} \phi_*(a_1) = a_1 - 4b_{3,2} - 3c_{2,1} + 3d_4 & \phi_*(a_2) = a_2 - 3d_2 & \phi_*(a_3) = a_3 - 3d_1 \\ \phi_*(a_4) = a_4 - 2b_{4,3} + c_{3,2} - 2d_5 & \phi_*(a_5) = a_5 + 2d_3 & \phi_*(a_6) = a_6 + 2d_2 \\ \phi_*(a_7) = a_7 & \phi_*(a_8) = a_8 & \end{cases} \tag{6.3}$$

6.3. **Calculations for the pair** $(\mathfrak{D}_1, \mathfrak{D}_2)$**.** In this part we provide the last two steps of the calculations.

Step 1 – We have isomorphisms

$$W(\mathfrak{D}_1) \simeq W(D_1) \oplus W(A \cdot D_{\widehat{P}_2}) \oplus W(AfA \cdot D_{\widehat{P}_2}), \quad W(\mathfrak{D}_2) \simeq W(D_{\widehat{P}_1}) \oplus W(AD_2) \oplus W(AfA \cdot D_{\widehat{P}_2}).$$

We transport the bases of $W(D_2)$ and $W(D_{\widehat{P}_2})$ by A and $A\phi A$. Therefore, if we take algebraic bases of $W(D_{\widehat{P}_1})$, $W(D_{\widehat{P}_2})$ and $W(D_2)$, the matrix of $\psi_* \ : \ W(\mathfrak{D}_1) \longrightarrow W(\mathfrak{D}_2)$ has the form

	$A\phi \cdot (\mathfrak{l}_i)_{1\leq i\leq 10}$	$A\phi \cdot (A\mathfrak{m}_i)_{1\leq i\leq 10}$	$A\phi \cdot (A\phi A\mathfrak{m}_i)_{1\leq i\leq 10}$
$(\mathfrak{l}_i)_{1\leq i\leq 10}$	$0_{10\times 10}$	$0_{10\times 10}$	$\mathscr{Q}$
$(A\mathfrak{m}_i)_{1\leq i\leq 10}$ $\mathrm{H}^0(\mathbb{P}^2, \mathrm{T}\mathbb{P}^2(5A\Delta))^\dagger$	$\mathscr{L}\mathscr{K}^{-1}$	$0_{55\times 10}$	$0_{55\times 10}$
$(A\phi A\mathfrak{m}_i)_{1\leq i\leq 10}$	$0_{10\times 10}$	$\mathrm{id}_{10\times 10}$	$0_{10\times 10}$

– Matrix $\mathscr{Y}$ *–*

The matrix $\mathscr{Q}$ is the 10×10 matrix given explicitly by

$$\begin{pmatrix}
-4\lambda_1 - 1 & q_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 4\mu_1 - 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 - 25\lambda_2 & \mu_1(25\lambda_2 - 8) & 6\lambda_2 - 1 & q_{3,4} & 0 & 0 & 0 & 0 & 0 & 0 \\
-25 & 25\mu_1 & 6 & 1 - 6\mu_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-50\lambda_3 & 8 + 50\lambda_3\mu_1 & 4(1 + 3\lambda_3) & -4\mu_2(1 + 3\lambda_3) & -1 & \mu_3 + \lambda_3 & 0 & 0 & 0 & 0 \\
-50 & 50\mu_1 & 12 & -12\mu_2 & 0 & 1 & 0 & 0 & 0 & 0 \\
q_{7,1} & q_{7,2} & 18\lambda_4 - 42 & \mu_2(42 - 18\lambda_4) & -10 & 10\mu_3 - 10 & -1 & \mu_4 + \lambda_4 & 0 & 0 \\
q_{8,1} & q_{8,2} & 18 & -18\mu_2 & 0 & 0 & 0 & 1 & 0 & 0 \\
q_{9,1} & q_{9,2} & q_{9,3} & q_{9,4} & -110 & 110\mu_3 - 62 & -12 & 12\mu_4 & -1 & q_{9,10} \\
q_{10,1} & q_{10,2} & 24 & -24\mu_2 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

where

$$\begin{aligned}
q_{1,2} &= \mu_1(1 + 4\lambda_1) - \lambda_1 \\
q_{3,4} &= \mu_2(1 - 6\lambda_2) + \lambda_2 \\
q_{7,1} &= 2\alpha + 200 + \lambda_4(3\mu_4^2 - \lambda_4^2 - 75 - 4\lambda_4) + 2\mu_4^3 \\
q_{7,2} &= 40 + \mu_1(75\lambda_4 - 2\alpha - 200 + 4\lambda_4^2 - 2\mu_4^3 - 3\mu_4^2\lambda_4 + \lambda_4^3) - (\mu_4 + \lambda_4)^2 \\
q_{8,1} &= 3(\mu_4^2 - \lambda_4^2) - 75 - 8\lambda_4 \\
q_{8,2} &= \mu_1(75 + 8\lambda_4 + 3(\lambda_4^2 - \mu_4^2)) - 2(\mu_4 + \lambda_4) \\
q_{9,1} &= 672 - 4\mu_5(\lambda_5 + 1) + 24(\mu_4^3 + \alpha) - 2(\lambda_5^2 + \mu_5^2) - 100\lambda_5 \\
q_{9,2} &= \mu_1(4\mu_5(1 + \lambda_5) - 24(\mu_4^3 + \alpha) + 2(\mu_5^2 + \lambda_5^2) + 100\lambda_5 - 672) + 320 - 2\alpha - 12\mu_4^2 \\
q_{9,3} &= 24\lambda_5 - 92 - 4\alpha \\
q_{9,4} &= \mu_2(92 + 4\alpha - 24\lambda_5) \\
q_{9,10} &= \mu_5 + 1 + \lambda_5, \quad q_{10,1} = -4(25 + \lambda_5 + \mu_5) \\
q_{10,2} &= 4\mu_1(25 + \lambda_5 + \mu_5).
\end{aligned}$$

Step 2 – We have $V(\mathfrak{D}_1) = \mathfrak{h}(\mathbb{P}^2)$ and $V(\mathfrak{D}_2) = \mathrm{H}^0(\mathbb{P}^2, \mathrm{T}\mathbb{P}^2(5A\Delta))$. Let us decompose a basis of $V(\mathfrak{D}_1)$ in an algebraic basis of $W(\mathfrak{D}_1)$:

	$(a_i)_{1\le i\le 8}$
$(\mathfrak{l}_i)_{1\le i\le 10}$	$\mathscr{M}_{\mathrm{a}}$
$(A\mathfrak{m}_i)_{1\le i\le 10}$	$\mathscr{M}'_{\mathrm{a}}$
$(A\phi A\mathfrak{m}_i)_{1\le i\le 10}$	$\mathscr{M}''_{\mathrm{a}}$

– *Matrix* $\mathscr{V}_1$ –

To compute quickly the matrices $\mathscr{M}'_{\mathrm{a}}$ and $\mathscr{M}''_{\mathrm{a}}$, we make the following remark (which can be proved by an easy computation): for any holomorphic vector field defined in a neighborhood $(0,0)$, if we lift it as a section of $W(D_{\widehat{P}_1})$ or $W(D_{\widehat{P}_2})$, this section depends only on the seven Taylor components ∂_y, $y\,\partial_y$, $z\,\partial_y$,∂_z, $y\,\partial_z$, $z\,\partial_z$, $y^2\,\partial_z$ of the vector field at $(0,0)$. The corresponding sections in algebraic bases are given by the tables

	∂_y	$y\,\partial_y$	$z\,\partial_y$	∂_z	$y\,\partial_z$	$z\,\partial_z$	$y^2\,\partial_z$
$\mathfrak{l}_1$	$-\lambda_1$	0	0	1	0	0	0
$\mathfrak{l}_2$	-1	0	0	0	0	0	0
$\mathfrak{l}_3$	2	0	0	0	$-\lambda_2$	0	0
$\mathfrak{l}_4$	0	0	0	0	-1	0	0
$\mathfrak{l}_5$	0	0	0	0	$-2\lambda_3$	0	0
$\mathfrak{l}_6$	0	0	0	0	-2	0	0
$\mathfrak{l}_7$	$-\lambda_4^2$	-3	0	λ_4^3	$-3\lambda_4$	2	0
$\mathfrak{l}_8$	$-2\lambda_4$	0	0	$3\lambda_4^2$	-3	0	0
$\mathfrak{l}_9$	0	0	-3	$2\lambda_5^2$	$-4\lambda_5$	0	-2
$\mathfrak{l}_{10}$	0	0	0	$4\lambda_5$	-4	0	0

– *Matrix* $\mathscr{Z}_1$ –

	∂_y	$y\,\partial_y$	$z\,\partial_y$	∂_z	$y\,\partial_z$	$z\,\partial_z$	$y^2\,\partial_z$
$\mathfrak{m}_1$	$-\mu_1$	0	0	1	0	0	0
$\mathfrak{m}_2$	-1	0	0	0	0	0	0
$\mathfrak{m}_3$	2	0	0	0	$-\mu_2$	0	0
$\mathfrak{m}_4$	0	0	0	0	-1	0	0
$\mathfrak{m}_5$	0	0	0	0	$-2\mu_3$	0	0
$\mathfrak{m}_6$	0	0	0	0	-2	0	0
$\mathfrak{m}_7$	$-\mu_4^2$	3	0	$-\mu_4^3$	$-3\mu_4$	-2	0
$\mathfrak{m}_8$	$-2\mu_4$	0	0	$-3\mu_4^2$	-3	0	0
$\mathfrak{m}_9$	0	0	3	$2\mu_5^2$	$-4\mu_5$	0	-2
$\mathfrak{m}_{10}$	0	0	0	$4\mu_5$	-4	0	0

– *Matrix* $\mathscr{Z}_2$ –

Therefore, in order to compute $\mathscr{M}'_{\mathrm{a}}$ and $\mathscr{M}''_{\mathrm{a}}$, it suffices to extract the seven aforementioned Taylor components of the vector field $A_*^{-1}a_i$ (resp. $(A\phi A)_*^{-1}a_i$) at $(0,0)$. Then we multiply the resulting vector by the matrix $\mathscr{Z}_1$ (resp. $\mathscr{Z}_2$) and we obtain $\mathscr{M}'_{\mathrm{a}}$ (resp. $\mathscr{M}''_{\mathrm{a}}$). We won't give the exact expressions of $\mathscr{M}'_{\mathrm{a}}$ and $\mathscr{M}''_{\mathrm{a}}$ because of lack of space.

We now deal with $V(\mathfrak{D}_2)$. We decompose a basis of vectors of $V(\mathfrak{D}_2)$ in an algebraic basis of $W(\mathfrak{D}_2)$. We get the following matrix

	$(Aa_i)_{1\leq i\leq 8}$	$(Ab_{p,q})_{0\leq p\leq q+1\leq m+1}$	$(Ac_{p,q})_{0\leq p\leq q+1\leq m+1}$	$(Ad_p)_{1\leq p\leq m}$
$(\mathfrak{l}_i)_{1\leq i\leq 10}$	$\mathscr{N}'_{\mathrm{a}}$	$\mathscr{N}'_{\mathrm{b}}$	$\mathscr{N}'_{\mathrm{c}}$	$\mathscr{N}'_{\mathrm{d}}$
$(A\mathfrak{m}_i)_{1\leq i\leq 10}$	$\mathscr{N}_{\mathrm{a}}$	$\mathscr{N}_{\mathrm{b}}$	$\mathscr{N}_{\mathrm{c}}$	$\mathscr{N}_{\mathrm{d}}$
$(Ab_{p,q})_{0\leq p\leq q+1\leq m+1}$	$0_{25\times 8}$	$\mathrm{id}_{25\times 25}$	$0_{25\times 25}$	$0_{25\times 25}$
$(Ac_{p,q})_{0\leq p\leq q+1\leq m+1}$	$0_{25\times 8}$	$0_{25\times 25}$	$\mathrm{id}_{25\times 25}$	$0_{25\times 25}$
$(Ad_p)_{1\leq p\leq m}$	$0_{5\times 8}$	$0_{5\times 25}$	$0_{5\times 25}$	$\mathrm{id}_{5\times 5}$
$(A\phi A\mathfrak{m}_i)_{1\leq i\leq 10}$	$\mathscr{N}''_{\mathrm{a}}$	$\mathscr{N}''_{\mathrm{b}}$	$\mathscr{N}''_{\mathrm{c}}$	$\mathscr{N}''_{\mathrm{d}}$

– *Matrix* $\mathscr{V}_2$ –

The matrices $\mathscr{N}'_{\mathrm{a}}$ (resp. $\mathscr{N}''_{\mathrm{a}}$), $\mathscr{N}'_{\mathrm{b}}$ (resp. $\mathscr{N}''_{\mathrm{b}}$), $\mathscr{N}'_{\mathrm{c}}$ (resp. $\mathscr{N}''_{\mathrm{c}}$) and $\mathscr{N}'_{\mathrm{d}}$ (resp. $\mathscr{N}''_{\mathrm{d}}$) appearing in $\mathscr{V}_2$ are computed in the same way as $\mathscr{N}'_{\mathrm{a}}$ (resp. $\mathscr{N}''_{\mathrm{a}}$).

6.4. **The result.** We can now state and prove the following result.

Theorem 6.1. *Let X be the rational surface obtained by blowing up the projective plane 15 times at the infinitely near points $\widehat{P}_1$, $A\widehat{P}_2$ and $A\phi A\widehat{P}_2$, and let ψ be the lift of $A\phi$ as an automorphism of X. Then the characteristic polynomial Q_ψ of the map ψ_* acting on the space $\mathrm{H}^1(X,\mathrm{T}X)$ of infinitesimal deformations of X is*

$$Q_\psi(x) = (x^2+3x+1)\,(x^2+18x+1)\,(x^2-7x+1)\,(x^2+x+1)\,(x-1)^2\,(x+1)^4\,(x^2-x+1)^4.$$

Besides, there is only one nontrivial Jordan block, which is a 2×2 Jordan block attached with the eigenvalue -1.

Proof. Let $\mathscr{E}_1$ be the subspace of $W(\mathfrak{D}_1)$ of dimension 22 defined by

$$\mathscr{E}_1 = \mathrm{Span}\left\{\begin{array}{c}\mathfrak{l}_1,\,\mathfrak{l}_2,\,\mathfrak{l}_3,\,\mathfrak{l}_4,\,\mathfrak{l}_5,\,\mathfrak{l}_6,\,\mathfrak{l}_7,\,\mathfrak{l}_8,\,\mathfrak{l}_9,\,\mathfrak{l}_{10},\\ A\mathfrak{m}_1,\,A\mathfrak{m}_2,\,A\mathfrak{m}_3,\,A\mathfrak{m}_4,\,A\mathfrak{m}_5,\,A\mathfrak{m}_6,\,A\mathfrak{m}_7,\\ A\phi A\mathfrak{m}_1,\,A\phi A\mathfrak{m}_2,\,A\phi A\mathfrak{m}_3,\,A\phi A\mathfrak{m}_4,\,A\phi A\mathfrak{m}_5\end{array}\right\}.$$

A direct calculation shows that $\mathscr{E}_1$ is a direct factor of $V(\mathfrak{D}_1)$ in $W(\mathfrak{D}_1)$. If $\mathscr{E}_2$ is the image of $\mathscr{E}_1$ in $W(\mathfrak{D}_2)$ by the natural injection, then $\mathscr{E}_2$ is a direct factor of $V(\mathfrak{D}_2)$ in $W(\mathfrak{D}_2)$. Therefore, the composition of the morphisms (6.2) can be expressed as

$$\mathscr{E}_1 \xrightarrow{\ j\ } \mathscr{E}_1\oplus V(\mathfrak{D}_1)\simeq W(\mathfrak{D}_1)\xrightarrow{\ \psi_*\ } W(\mathfrak{D}_2)\simeq \mathscr{E}_2\oplus V(\mathfrak{D}_2)\xrightarrow{\ p\ }\mathscr{E}_2\simeq\mathscr{E}_1\ .$$

The matrix of j (resp. p) can be computed using $\mathscr{V}_1$ (resp. $\mathscr{V}_2$) and the matrix of ψ_* is $\mathscr{Y}$. □

Corollary 6.2. $m(X,f)\leq 2$.

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CNRS, I2M (Marseille) & IHÉS
E-mail address: jgrivaux@math.cnrs.fr