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LOCI IN STRATA OF MEROMORPHIC QUADRATIC DIFFERENTIALS WITH FULLY DEGENERATE LYAPUNOV SPECTRUM

JULIEN GRIVAUX AND PASCAL HUBERT
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ABSTRACT. We construct explicit closed GL(2;ℝ)-invariant loci in strata of meromorphic quadratic differentials of arbitrarily large dimension with fully degenerate Lyapunov spectrum. This answers a question of Forni–Matheus–Zorich.

1. INTRODUCTION

Lyapunov exponents of the Teichmüller flow have been studied a lot since the work of Zorich ([14, 15]) and Forni [9]. Their understanding is important for applications to the dynamics of interval-exchange transformations and polygonal billiards. A big breakthrough is the Eskin–Kontsevich–Zorich formula for the sum of positive Lyapunov exponents [5]. Given a SL(2;ℝ)-invariant suborbifold of a stratum of quadratic differentials, they relate the sum $\lambda_1+\cdots+\lambda_g$ to the Siegel–Veech constant of the invariant locus[1].

By a theorem of Kontsevich and Forni, the sum $\lambda_1+\cdots+\lambda_g$ is also the integral over the invariant locus of the curvature of the Hodge bundle along Teichmüller disks ([9, 5]). Using this interpretation, every Lyapunov exponent is computed for cyclic covers of the sphere branched over 4 points ([4, 8], see also [2], and [13] for abelian covers). For some cyclic covers, Forni–Matheus–Zorich have remarked that the sum $\lambda_1+\cdots+\lambda_g$ is equal to zero [8, Thm. 35]. This surprising fact means that the complex structure of the underlying Riemann surface is constant along the Teichmüller disk. Forni–Matheus–Zorich ask whether it is possible to construct other invariant loci with this property (see [8, p. 312]). The content of this article is to give a simple explanation of the phenomenon discovered by Forni–Matheus–Zorich.

THEOREM 1. *There exist closed* GL(2;ℝ)*-invariant loci of quadratic differentials of arbitrarily large dimension with zero Lyapunov exponents.*

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[1]For quadratic differentials, two bundles can be considered. In this article, we will only be interested in the bundle with fiber $H^1(X,\mathbb{R})$ over a Riemann surface X. The Lyapunov exponents of this bundle are often denoted by $\lambda_1^+,\dots,\lambda_g^+$.

This result can be interpreted in the following way: the projection of such a locus to the moduli space of *compact* Riemann surfaces is a point. Remark that the situation for strata of *abelian* differentials is completely different: there are finitely many invariant suborbifolds with fully degenerate Lyapunov spectrum (meaning in this setting that all exponents are zero except λ_1 which is 1), and they are arithmetic Teichmüller curves (see [12, 10, 8] and [1]).

2. BACKGROUND MATERIAL

2.1. **The Teichmüller flow for translation surfaces.** A translation surface is a pair (X,ω), where X is a compact Riemann surface and ω is a holomorphic one-form on X. If $S(\omega)$ is the set of the zeroes of ω, there exists an open covering of $\tilde{X} = X \setminus S(\omega)$ and holomorphic charts $\varphi_i \colon U_i \to X$ such that $\varphi_i^* \omega = dz$ for all i. For such an atlas, the transition functions are translations. The form ω induces a flat metric $|\omega^2|$ on the open surface $\tilde{X}$, whose area is the integral $\frac{i}{2}\int_X \omega \wedge \bar{\omega}$. We could have taken meromorphic 1-forms instead of holomorphic ones, but in that case the area of the surface is never finite.

There is a natural action of $\mathrm{GL}(2,\mathbb{R})$ on translation surfaces given as follows: first we pick an atlas of charts of $\tilde{X}$, where all transitions functions are translations by some complex vectors v_{ij}, which we will consider as vectors in $\mathbb{R}^2$. Then, for any M in $\mathrm{GL}(2,\mathbb{R})$, we get an open surface $\tilde{X}_M$ defined by an atlas whose transition functions are translations by $M.v_{ij}$. This surface is diffeomorphic to $\tilde{X}_M$. Therefore, we can fill the holes and extend the complex structure in a unique way: the result is a compact Riemann surface X_M diffeomorphic to X endowed with a meromorphic differential ω_M of finite volume, hence holomorphic. The action of $\mathrm{GL}(2;\mathbb{R})$ is defined by the formula $M.(X,\omega) = (X_M,\omega_M)$. The action of $\mathrm{SL}(2;\mathbb{R})$ preserves the volume of translation surfaces.

The subgroup of $\mathrm{SL}(2;\mathbb{R})$ of matrices M such that $M.(X,\omega) = (X,\omega)$ up to diffeomorphism is called the Veech group of (X,ω). If $M_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ then the curve $(X_t,\omega_t) := M_t.(X,\omega)$ is called the orbit under the Teichmüller flow of (X,ω). If M_θ is a rotation in $\mathrm{SO}(2,\mathbb{R})$, then $M_\theta.(X,\omega) = (X, e^{i\theta}\omega)$.

If we fix multiplicities $(m_1,\ldots,m_r)$ such that $\sum_{i=1}^r m_i = 2g-2$, the stratum of translation surfaces $\mathscr{H}(m_1,\ldots,m_r)$ is the set of translations surfaces (X,ω), where ω has r distinct zeroes of multiplicities $m_1,\ldots,m_r$ modulo diffeomorphism. The normalized stratum $\mathscr{H}_1(m_1,\ldots,m_r)$ is the locus of flat surfaces with unit area in $\mathscr{H}(m_1,\ldots,m_r)$, and the projective stratum $\mathrm{P}\mathscr{H}(m_1,\ldots,m_r)$ is obtained by taking the quotient of $\mathscr{H}(m_1,\ldots,m_r)$ under the natural $\mathbb{C}^\times$-action on forms. Strata and projective strata are complex orbifolds of respective dimensions $2g+r-1$ and $2g+r-2$ if $g \geq 2$.

There are standard coordinates on the stratum $\mathscr{H}(m_1,\ldots,m_r)$, called period coordinates. Fix (X,ω) in this stratum, and let $A_1,\ldots,A_g,\mathrm{B}_1,\ldots,\mathrm{B}_g$ be a symplectic basis of $\mathrm{H}_1(X,\mathbb{Z})$ and $C_1,\ldots,C_{r-1}$ be $r-1$ paths joining a zero of ω to all the

$r-1$ other zeroes. The map

$$(X,\omega) \to \left(\int_{A_1} \omega, \ldots, \int_{A_g} \omega, \int_{B_1} \omega, \ldots, \int_{B_g} \omega, \int_{C_1} \omega, \ldots, \int_{C_{r-1}} \omega\right)$$

yields an orbifold chart on $\mathscr{H}(m_1, \ldots, m_r)$. These charts allow to define a canonical volume element on $\mathscr{H}(m_1, \ldots, m_r)$, $\mathscr{H}_1(m_1, \ldots, m_r)$, and $\mathrm{P}\mathscr{H}(m_1, \ldots, m_r)$. By classical results of Masur and Veech, connected components of projective strata have finite volume.

Let $\mathbb{H} = \mathrm{SL}(2;\mathbb{R})/\mathrm{SO}(2)$ denote the Poincaré upper half-plane. For any (X,ω) in a projective stratum, the $\mathrm{SL}(2;\mathbb{R})$-action factorizes to a holomorphic map

$$\mathbb{H} \to \mathrm{P}\mathscr{H}(m_1, \ldots, m_r)$$

which is an immersion. The image of this map is called a Teichmüller disk, it is stable under the Teichmüller flow. Besides, Teichmüller disks induce a smooth foliation with holomorphic leaves on $\mathrm{P}\mathscr{H}(m_1, \ldots, m_r)$.

Assume that the Veech group Γ of (X,ω) is a lattice in $\mathrm{SL}(2;\mathbb{R})$. Then the image $\mathbb{H}/\Gamma$ of the corresponding Teichmüller disk in the projective stratum is called a Teichmüller curve.

All these considerations generalize to the so-called half-translation surfaces, which are pairs (X, q), where q is a quadratic holomorphic (for the time being) differential on X. The transition functions of a well-chosen atlas of charts on the open surface are half translations, that is either translations or flips. The area of the flat metric on $\tilde{X}$ is $\frac{1}{2}\int_X |q|$, and we still have an action of $\mathrm{GL}(2,\mathbb{R})$ as well as a Teichmüller flow. The period coordinates on strata of quadratic differentials are obtained as follows: for any (X, q) in a stratum, we take the twofold branched covering $\mathfrak{p}\colon \widehat{X} \to X$ given by the holonomy representation of q, which is given by a morphism from $\pi_1(X)$ to $\mathbb{Z}/2\mathbb{Z}$. Let j be the corresponding involution acting on $\widehat{X}$. The quadratic differential $\mathfrak{p}^*q$ is the square of an abelian differential ω. The period coordinates of (X, q) are obtained by taking j-anti-invariant absolute and relative periods of (X, q).

However, a major difference happens for quadratic differentials: it is possible to take meromorphic quadratic differentials and still get finite volume for the corresponding flat surface. More precisely, (X, q) has finite volume if and only if q has poles of order at most one. Therefore we have strata, normalized strata and projective strata $\mathscr{Q}(m_1, \ldots, m_r)$, $\mathscr{Q}_1(m_1, \ldots, m_r)$ and $\mathrm{P}\mathscr{Q}(m_1, \ldots, m_r)$, where $\sum_{i=1}^r m_i = 4g-4$ and each m_i is either positive or equal to -1.

Let S be a finite subset of X of cardinal n, so that (X, S) gives a point in the marked Teichmüller space $\mathscr{T}_{g,n}$ (genus g with n marked points). The cotangent space of $\mathscr{T}_{g,n}$ at X is exactly the space $\mathscr{Q}_S(X)$ of holomorphic quadratic differentials on $X \smallsetminus S$ with poles of order one at some points of S. There is a norm on $\mathscr{Q}_S(X)$ given by $\|q\| = \int_X |q|$, as well as a dual norm on $\mathscr{Q}_S(X)^*$. The corresponding distance on $\mathscr{T}_{g,n}$ is the Teichmüller metric.

Let us fix (X,S) as well as an element q in $\mathcal{Q}_S(X)$. There is a complex linear form μ_q on $\mathcal{Q}_S(X)$ given by scalar product with the L^∞ Beltrami differential $\dfrac{|q|}{q}$:

$$\mu_q(\tilde{q}) = \int_X \tilde{q}\,\frac{|q|}{q}.$$

Note that $\mu_q(q) = \int_X |q| > 0$ so that μ_q is nonzero. Besides, we have $\|\mu_q\| = 1$. The map $q \to \mu_q$ gives a nonlinear isomorphism between the unit spheres of $\mathcal{Q}_S(X)$ and $\mathcal{Q}_S(X)^*$, hence between the unitary cotangent space $\mathrm{U}^*\mathcal{T}_{g,n}$ and the unitary tangent space $\mathrm{U}\mathcal{T}_{g,n}$.

If (X,q) is given and S is the set of poles of q, the Teichmüller flow of (X,q) introduced formerly is the geodesic flow (for the Teichmüller metric) on $\mathcal{T}_{g,n}$ starting from X in the direction μ_q.

2.2. **The period mapping.** For any compact Riemann surface X, $\mathrm{H}^1(X,\mathbb{C})$ is the orthogonal sum (for the intersection form) of $\Omega(X)$ and $\overline{\Omega(X)}$. Besides, the composition

$$\psi\colon \mathrm{H}^1(X,\mathbb{R}) \to \mathrm{H}^1(X,\mathbb{C}) \xrightarrow{\mathrm{pr}_1} \Omega(X)$$

is an isomorphism. The Hodge norm $\|\ \|_{\mathrm{Hodge}}$ is the unique norm on $\mathrm{H}^1(X,\mathbb{R})$ making ψ an isometry.

Let us now consider a local holomorphic family of curves, that is a proper holomorphic submersion $\pi\colon \mathfrak{X} \to \mathcal{B}$ whose fibers $(X_b)_{b\in\mathcal{B}}$ are compact Riemann surfaces of some genus g, where $\mathcal{B}$ is a small ball in $\mathbb{C}^n$. The Hodge bundle $\mathcal{H}$ is a holomorphic vector bundle on $\mathcal{B}$ of rank g whose fiber at each point b is the vector space $\mathcal{H}_b := \Omega(X_b)$. The local system $\mathrm{R}^1\pi_*\mathbb{R}_{\mathfrak{X}}$ is trivial, which means that we can canonically identify all the vector spaces $\mathrm{H}^1(X_b,\mathbb{R})$ to some fixed real vector space $\mathbb{V}$ of dimension $2g$. The local period map

$$\xi\colon \mathcal{B} \to \mathrm{Gr}(g,\mathbb{V}^{\mathbb{C}})$$

associates to any b the subspace $\mathcal{H}_b$ in the Grassmannian of g-dimensional complex subspaces of $\mathbb{V}^{\mathbb{C}}$. The derivative of ξ at a point b in $\mathcal{B}$ is a linear map from $\mathrm{T}_b^{1,0}\mathcal{B}$ to $\mathrm{End}(\mathcal{H}_b, \mathbb{V}^{\mathbb{C}}/\mathcal{H}_b)$, which is isomorphic to $\mathrm{End}(\mathcal{H}_b, \overline{\mathcal{H}_b})$.

The differential of ξ can be explicitly computed: ξ induces a classifying map $\xi_{\mathrm{Teich}}\colon \mathcal{B} \to \mathcal{T}_g$. Then we have the following formula due to Ahlfors: for any vector v in $\mathrm{T}_b\mathcal{B}$ and any elements α and β in $\Omega(X_b)$,

$$(\beta, \overline{\xi'_v(\alpha)}) = \int_X \alpha \otimes \beta.\, \xi'_{\mathrm{Teich}}(v).$$

In this formula, $\xi'_{\mathrm{Teich}}(v)$ is a tangent vector to $\mathcal{T}_g$, hence represented by a Beltrami differential which is a tensor field on X of type $(-1,1)$. Thus, the integrand in the above formula of type $(2,0)+(-1,1) = (1,1)$. We can also think of $\xi'_{\mathrm{Teich}}(v)$ as a linear form on $\mathcal{Q}(X)$; in this case the above formula reads

$$(\beta, \overline{\xi'_v(\alpha)}) = \xi'_{\mathrm{Teich}}(v)\,\{\alpha \otimes \beta\}.$$

It is possible to give another interpretation on ξ'. For this we consider the exact sequence of holomorphic vector bundles

$$0 \to \mathscr{H} \to \mathbb{V} \otimes \mathscr{O}_{\mathscr{B}} \to \overline{\mathscr{H}} \to 0.$$

The bundle $\mathbb{V} \otimes \mathscr{O}_{\mathscr{B}}$ carries a natural flat connection (the Gauß–Manin connection), but $\mathscr{H}$ is not in general a flat subbundle of $\mathbb{V} \otimes \mathscr{O}_{\mathscr{B}}$. A precise way to measure this (see formula (2) below) is the second fundamental form σ associated with this exact sequence and the Gauß–Manin connection; it is a $(1,0)$-form with values in $\mathrm{Hom}(\mathscr{H}, \overline{\mathscr{H}})$. A simple calculation shows that

$$\sigma = \xi'. \tag{1}$$

The Hodge bundle $\mathscr{H}$ carries a natural metric given by the intersection form, its curvature form is given by the formula

$$\Theta_{\mathscr{H}} = \sigma^* \wedge \sigma. \tag{2}$$

By "$\wedge$" we mean composition on the fiber and wedge-product on the base. In particular, $i\,\mathrm{Tr}\Theta_{\mathscr{H}}$ is a positive $(1,1)$-form on $\mathscr{B}$.

For any compact half-translation surface (X, q), Forni's B-form is a bilinear form on $\Omega(X)$ defined by

$$\mathrm{B}_q(\alpha, \beta) = \int_X \alpha \otimes \beta \frac{|q|}{q} \cdot$$

If $\xi'_{\mathrm{Teich}}(v)$ has unit norm, we can write it as μ_q for some holomorphic quadratic differential on X. Then we have $(\beta, \overline{\xi'_v(\alpha)}) = \mathrm{B}_q(\alpha, \beta)$. In the case of a Teichmüller orbit (X_t, q_t), if we differentiate along the vector field $\frac{\partial}{\partial_t}$, we get the formula

$$(\beta, \overline{\xi'_{\partial_t}(\alpha)}) = \mathrm{B}_{q_t}(\alpha, \beta). \tag{3}$$

Applying Cauchy–Schwarz inequality, $|\mathrm{B}_q(\alpha, \beta)| \le \|\alpha\| \times \|\beta\|$ with equality if and only if there exists a holomorphic one-form ω and two complex constants c and c' such that $q = \omega^2$, $\alpha = c\omega$ and $\beta = c'\omega$. In particular, if q is meromorphic with simples poles, $|||\mathrm{B}_q||| < 1$, where $|||\,.\,|||$ denotes the operator norm.

We recall now Forni's inequality: let (X, q) be a half-translation surface, (X_t, q_t) be its orbit under the Teichmüller flow, v be in $\mathrm{H}^1(X, \mathbb{R})$ and $t \to v_t$ be its parallel transport under the Teichmüller flow for the Gauß–Manin connection. We write $v_t = \chi_t + \overline{\chi_t}$, where χ_t is in $\Omega(X_t)$. Then a simple calculation gives

$$\partial_t \|v_t\|_{\mathrm{Hodge}} = \mathrm{B}_{q_t}(\chi_t, \chi_t).$$

Combined with the inequality $|||B||| \le 1$, this gives Forni's inequality

$$\left| \partial_t \{\log \|v_t\|_{\mathrm{Hodge}}\} \right| \le 1. \tag{4}$$

2.3. **Lyapunov exponents of the KZ cocycle.** The parallel transport for the Gauß–Manin connection of vectors of $\mathrm{H}^1(X,\mathbb{R})$ under the Teichmüller flow is called the Kontsevich–Zorich cocycle. Recall that the Teichmüller flow is ergodic on every connected component $\mathfrak{D}_1$ of the normalized stratum $\mathcal{Q}_1(m_1,\ldots,m_r)$. By Oseledets' theorem, it is possible to associate $2g$ Lyapunov exponents to this cocycle.

Forni's inequality (4) implies that the KZ cocycle is log-integrable, so that the Lyapunov exponents are well-defined. Since the cocycle is symplectic, the Lyapunov spectrum is of the form $\{-\lambda_1,-\lambda_2,\ldots,-\lambda_g,\lambda_g,\lambda_{g-1},\ldots,\lambda_1\}$, where $\lambda_1\geq\ldots\geq\lambda_g$.

Note that the exponents λ_i are called λ_i^+ in numerous papers (e.g., in [5, 8]). The exponents λ_i^- will never be considered in the article.

By (4), all λ_i's are at most one. If the component $\mathfrak{D}_1$ is orientable, which means that every quadratic differential occurring in the stratum is the square of an abelian differential, then the top Lyapunov exponent λ_1 equals one. If not, the norm of Forni's B form is strictly smaller than one so that $\lambda_1<1$.

For any (X,q) in a stratum $\mathrm{P}\mathcal{Q}(m_1,\ldots,m_r)$, the Poincaré metric on $\mathbb{H}$ induces a metric on the Teichmüller disk passing through (X,q). The corresponding volume element defines a relative $(1,1)$ form $\mathrm{dV}_{\mathrm{Teich}}$, where by "relative" we mean relative with respect to the foliation by Teichmüller disks. If Θ is the curvature of the Hodge bundle on $\mathrm{P}\mathcal{Q}(m_1,\ldots,m_r)$, its trace is also a relative $(1,1)$ form on the projective stratum. Let $\Lambda\colon \mathrm{P}\mathcal{Q}(m_1,\ldots,m_r)\to\mathbb{R}$ be defined by the formula

$$\Lambda=\frac{\mathrm{Tr}\,\Theta}{\mathrm{dV}_{\mathrm{Teich}}}\cdot$$

Then Kontsevich–Forni's main formula for the Lyapunov exponents is

$$\lambda_1+\ldots+\lambda_g=\int_{\mathfrak{D}}\Lambda(X,q)\,\mathrm{dV},$$

where $\mathfrak{D}$ is the projection of $\mathfrak{D}_1$ in the projective stratum and dV is the normalized volume element on $\mathfrak{D}$ of total mass one. For any (X,q) in $\mathfrak{D}$, let $\theta_1,\ldots,\theta_g$ be the eigenvalues of Forni's B-form in the direction of the Teichmüller flow when diagonalized in an orthonormal basis for the intersection form. Using formulæ (2), (1) and (3), we see that

$$\lambda_1+\ldots+\lambda_g=\int_{\mathfrak{D}}\left\{|\theta_1(X,q)|^2+\ldots+|\theta_g(X,q)|^2\right\}\mathrm{dV}. \tag{5}$$

Forni's inequality implies that $|\theta_i(X,q)|\leq 1$ for all i so that $\lambda_1+\ldots+\lambda_g\leq g$.

Thanks to the main result of [6], any closed SL(2;$\mathbb{R}$)-invariant locus in the projective stratum $\mathrm{P}\mathcal{Q}(m_1,\ldots,m_r)$ is affine in period coordinates, hence carries a natural SL(2;$\mathbb{R}$)-invariant probability measure. It is also possible to define Lyapunov exponents for this measure, and formula (5) holds.

If (X,q) is any half-translation surface, the closure of its SL(2;$\mathbb{R}$)-orbit in the normalized stratum is affine in period coordinates. It follows from [3] that almost every direction θ, the real Teichmüller flow of $(X,e^{i\theta}q)$ is Oseledets-generic for the corresponding natural probability measure. Therefore it makes sense to consider Lyapunov exponents of (X,q), and formula (5) is still valid if we integrate on the closure of the $\mathrm{PGL}(2;\mathbb{R})$-orbit.

3. THE FORNI LOCUS

3.1. **General properties.** Let $\mathfrak{D}$ be a connected component of the projective stratum $\mathrm{P}\mathcal{Q}(m_1,\dots,m_r)$.

DEFINITION 1. The Forni locus of $\mathfrak{D}$ is the set of elements (X,q) in $\mathfrak{D}$ such that for all holomorphic 1-forms α and β on X, $\mathrm{B}_q(\alpha,\beta)=0$.

REMARK 1. In the terminology used in [10], the Forni locus is called the rank 0 locus.

Let us now recall Noether's Theorem (see [7, p. 104 & 159]):

PROPOSITION 1. *Let X be a compact Riemann surface of genus g and*

$$\tau\colon \mathrm{Sym}^2\,\Omega^1(X)\to\mathcal{Q}(X)$$

be the multiplication map.

(i) *If X is not hyperelliptic or if $g\le 2$, τ is surjective.*
(ii) *If X is hyperelliptic,* $\mathrm{Im}(\tau)$ *has codimension $g-2$ in $\mathcal{Q}(X)$ and consists of the quadratic differentials invariant under the hyperelliptic involution.*

Since τ is the transpose of the derivative of the period map, Noether's result has the following geometric interpretation:

PROPOSITION 2 (Infinitesimal Torelli's theorem)**.** *Let $\xi\colon\mathcal{T}_g\to\mathbb{H}_g$ be the period map. Then ξ is an immersion outside the hyperelliptic locus or everywhere if $g\le 2$, and the restriction of ξ to the hyperelliptic locus is also an immersion.*

Remark that Forni's B-form factors through $\mathrm{Im}\,\tau$, and can be extended naturally to $\mathcal{Q}(X)$ by the formula $\mathrm{B}_q(\tilde q)=\int_X \tilde q\,\frac{|q|}{q}$.

The key proposition of this section is:

PROPOSITION 3. *Let (X,q) be a half-translation surface, n the number of poles of q, and $\mathbb{D}$ be its Teichmüller disk. Then the following are equivalent:*

(i) *$\mathbb{D}$ lies in the Forni locus.*
(ii) *The forgetful map $\mathcal{T}_{g,n}\to\mathcal{T}_g$ maps $\mathbb{D}$ to a point.*
(iii) *For any (X_t,q_t) in $\mathbb{D}$, the extension of B_{q_t} to $\mathcal{Q}(X_t)$ vanishes.*
(iv) *All Lyapunov exponents of (X,q) are zero.*

Proof. (i) $\Rightarrow$ (ii) Using (3), the composite map $\mathbb{D}\hookrightarrow\mathcal{T}_{g,n}\to\mathcal{T}_g\xrightarrow{\xi}\mathbb{H}_g$ has zero derivative. Assume that $\mathbb{D}$ is not contained in the hyperelliptic locus. Thanks to the infinitesimal Torelli theorem, $\mathbb{D}$ is mapped to a point via the forgetful map $\mathcal{T}_{g,n}\to\mathcal{T}_g$. Assume now that $\mathbb{D}$ is contained in the hyperelliptic locus. Then the restriction of ξ to this locus is again an immersion, and we can apply the same argument.

(ii) ⇒ (iii) If (X_t, q_t) is a point in $\mathbb{D}$, the derivative of projection of the Teichmüller flow of (X_t, q_t) on $\mathcal{T}_g$ is the linear form $\tilde{q} \to \mathrm{B}_{q_t}(\tilde{q})$ on $\mathcal{Q}(X_t)$.

(iii) ⇒ (i) Obvious.

(i) ⇔ (iv) Let $\mathcal{V}$ be the closure of the PSL(2;ℝ)-orbit of X and ν the corresponding PSL(2;ℝ)-invariant probability measure. If $\lambda_1, \ldots, \lambda_g$ are the Lyapunov exponents of (X, q), then

$$\lambda_1 + \ldots + \lambda_g = \int_{\mathcal{V}} \left\{\theta_1(X,q) + \ldots + \theta_g(X,q)\right\} d\nu.$$

Since all θ_i's are nonnegative and continuous functions, $\lambda_1 = \ldots = \lambda_g = 0$ if and only if all θ_i's vanish on $\mathbb{D}$. □

COROLLARY 1. *If q is a holomorphic quadratic differential on X, the Teichmüller disk of (X, q) is not included in the Forni locus. In particular some Lyapunov exponents are nonzero.*

Proof. If q is holomorphic, $\mathrm{B}_q(q) > 0$ and we apply Proposition 3. □

By Proposition 3 the vanishing along a Teichmüller disk of the B-form and of its extension to the quadratic differentials are equivalent. This is no longer the case if we consider these properties for a single half-translation surface (X, q), as shown by the example below.

Let X be a hyperelliptic surface of genus at least 3, let j be the hyperelliptic involution, and let q be an anti-invariant holomorphic quadratic differential (if X is the Riemann surface of a polynomial $w^2 - P(z)$, we can take $q = w^{-1} dz^{\otimes 2}$). Since any holomorphic 1-form on X is anti-invariant under j^*, $\mathrm{B}_q = 0$. Hence we see that the following properties are satisfied:

- q is holomorphic.
- (X, q) is in the Forni locus.
- The extension of B_q to $\mathcal{Q}(X)$ is nonzero since $\mathrm{B}_q(q) > 0$.
- The Teichmüller disk of (X, q) is not contained in the Forni locus (see Corollary 1).

This also gives a concrete example showing that the Forni locus is not SL(2;ℝ)-invariant.

3.2. **Galois coverings and pillow-tiled surfaces.** Let (X, q) be a half-translation surface and (Y, π) be an arbitrary finite covering of X with branching locus S. Assume that for any point y in Y above S, the ramification index of π at y is at least 2. Then $\pi^* q$ is holomorphic, so that $\mathrm{B}_{\pi^* q}$ is nonzero on $\mathcal{Q}(Y)$. Thanks to Corollary 1, the Teichmüller disk of $(Y, \pi^* q)$ doesn't lie in the Forni locus. Using this observation, we have the following result:

PROPOSITION 4. *Let (X, q) be a half-translation surface and (Y, π) be a finite Galois covering of X with branch locus S. If the Teichmüller disk of $(Y, \pi^* q)$ lies in the Forni locus, then at least one pole of q does not belong to S.*

As a particular by-product, we get:

COROLLARY 2. *Let (X,q,π) be a pillow-tiled surface such that π is Galois. Then the Teichmüller disk of (X,q) lies in the Forni locus if and only if the branching locus of π contains at most three points.*

Proof. Let $q_{\rm st}$ be the standard meromorphic differential on $\mathbb{P}^1$ with four simple poles such that $q=\pi^* q_{\rm st}$. By definition the branching locus of π lies in the set of poles of $q_{\rm st}$. If X is in the Forni locus, according to Proposition 4, one of the poles of $q_{\rm st}$ is not a branching point of π.

Conversely, assume that the branching locus of π has less than four points. If $\{z_1,z_2,z_3,z_4\}$ are the four poles of $q_{\rm st}$, let us assume that z_4 is not a branch point of π. The complex Teichmüller flow of $(\mathbb{P}^1,q_{\rm st})$ is of the form $(\mathbb{P}^1,q_t)$, where q_t has poles at z_1, z_2, z_3 and another point $z_4(t)$ such that $[z_1,z_2,z_3,z_4(t)]=t$. Let $\mathring{X}$ be the *open* Riemann surface obtained by removing $\pi^{-1}\{z_1,z_2,z_3\}$. Then $\mathring{X}$ is an unramified covering of $\mathbb{P}^1\smallsetminus\{z_1,z_2,z_3\}$. It follows that $(\mathring{X}\smallsetminus\pi^{-1}(z_4(t)),\pi^*q_t)$ parametrizes the Teichmüller disk of (X,q) in $\mathrm{T}^*\mathscr{T}_{g,n}$ (where n is the number of poles of q). This disk maps to $\{X\}$ via the forgetful map $\mathscr{T}_{g,n}\to\mathscr{T}_g$. Thanks to Proposition 3, the Teichmüller disk of (X,q) lies in the Forni locus. □

Let us now consider pillow-tiled surfaces arising as *cyclic* coverings of the projective line. They are given by a combinatorial datum (N,a_1,a_2,a_3,a_4), where $0<a_i\le N$, $\gcd(a_1,a_2,a_3,a_4,N)=1$ and $\sum_{i=1}^4 a_i\equiv 0\,(N)$: the associated cyclic covering is the Riemann surface of the polynomial

$$w^N-(z-z_1)^{a_1}(z-z_2)^{a_2}(z-z_3)^{a_3}(z-z_4)^{a_4}.$$

In topological terms, if $(\gamma_i)_{1\le i\le 4}$ are small loops around the z_i's for $1\le i\le 4$, then the kernel of the group morphism

$$\pi_1(\mathbb{P}^1\smallsetminus\{z_1,z_2,z_3,z_4\})\to\mathbb{Z}/N\mathbb{Z}$$

given by $\gamma_i\to a_i$ defines a true cyclic covering of $\mathbb{P}^1\smallsetminus\{z_1,z_2,z_3,z_4\}$ of degree N, which extends to a branched cyclic covering of the projective line.

In [8, Thm. 35], the authors prove that all Lyapunov exponents of the Teichmüller curve corresponding to a cyclic covering are 0 if one of the integers a_i equals N.

PROPOSITION 5. *If (X,q) is a pillow-tiled surface obtained by a cyclic covering of $\mathbb{P}^1$ with combinatorial datum (N,a_1,a_2,a_3,a_4), then the Teichmüller disk of (X,q) lies in the Forni locus if and only if one of the a_i's equals N.*

Proof. Thanks to Corollary 2, it suffices to prove that the projection π of the covering is branched at three points or less if and only if one of the a_i's equals N. If $\{z_1,z_2,z_3,z_4\}$ are the four points defining the cyclic cover, the ramification index of π at any point of $\pi^{-1}(z_i)$ is $\frac{N}{\gcd(N,a_i)}$. □

3.3. **Construction of Forni subspaces.** In this section, we provide the precise statement underlying Theorem 1 as well as its proof.

Let $m_1,\dots,m_r$ and k be positive integers such that $(\sum_{i=1}^r m_i)-k=-4$, and let $\mathscr{S}$ be the set of couples $(q,x_1,\dots,x_{k-3})$ such that q is a meromorphic differential on $\mathbb{P}^1$ with simple poles at 0, 1, ∞ and the x_i's, and q has r zeroes of order $m_1,\dots,m_r$. It is a smooth $\mathrm{GL}(2;\mathbb{R})$-invariant submanifold of $\mathrm{T}^*\mathscr{M}_{0,[k]}$ (where the bracket means that the points are ordered).

Let us fix a covering (Y,π) of $\mathbb{P}^1$ ramified over 0, 1 and ∞, and let g be the genus of Y. Put

$$n = \#\left\{y \in \pi^{-1}\{0,1,\infty\} \text{ such that } \pi \text{ is unramified at } y\right\} + \deg(\pi) \times (k-3)$$

We have a natural map

$$\chi\colon \mathscr{S} \to \mathrm{T}^*_{\mathrm{orb}}\mathscr{M}_{g,n}$$

given by $\chi(q) = (Y,\pi^* q)$, where $\mathrm{T}^*_{\mathrm{orb}}$ denotes the *orbifold* cotangent bundle.

THEOREM 2. *Let $\mathscr{W}$ be the image of χ.*

1. *The map $\chi\colon \mathscr{S} \to \mathscr{W}$ is a holomorphic orbifold map, which is a local immersion. Besides, $\mathscr{W}$ is a suborbifold*[2] *of the orbifold cotangent bundle of $\mathscr{M}_{g,n}$ of dimension $r+k-2$.*
2. *$\mathscr{W}$ is $\mathrm{GL}(2;\mathbb{R})$-invariant and lies in the Forni locus, and the projection of $\mathscr{W}$ by the map $\mathrm{T}^*_{\mathrm{orb}}\mathscr{M}_{g,n,} \to \mathscr{M}_{g,n} \to \mathscr{M}_g$ is $\{Y\}$.*
3. *The Lyapunov spectrum of $\mathscr{W}$ is fully degenerate.*

Proof. Let q be a point in $\mathscr{S}$, and U be a small neighborhood of q in $\mathscr{S}$. It is possible to lift locally χ to a smooth map $\widehat{\chi}$ from U to $\mathrm{T}^*\mathscr{T}_{g,n}$, so that χ is a smooth orbifold map.

If q_1, q_2 are two elements in U such that $\chi(q_1) = \chi(q_2)$, then there exists φ in $\mathrm{Aut}(Y)$ such that $\varphi^*(Y,\pi^* q_1) = (Y,\pi^* q_2)$. Thus the fibers of $\widehat{\chi}_{|U}$ are finite. But $\widehat{\chi}$ is affine in period coordinates, so that it is an immersion on U.

Point (2) is proved using the same argument as in Corollary 2, which corresponds to the particular case $r = 0$.

Lastly, the fact that the Lyapunov spectrum of $\mathscr{W}$ is totally degenerate results from the implication (ii) $\Rightarrow$ (iv) in Proposition 3. □

REMARK 2. In view of the existing terminology in the abelian case, it is natural to call $\mathscr{W}$ a *Forni subspace* of the stratum.

4. NUMERICAL CONSTRAINTS

In this section, given a half-translation surface (X,q) whose Teichmüller disk lies in the Forni locus, we give some numerical constraints relating the genus g of X, the number of n of simple poles of q and the degree d when (X,q) is a pillow-tiled surface.

PROPOSITION 6. *Let (X,q) be a half-translation surface of genus $g \geq 1$ whose Teichmüller disk lies in the Forni locus, and let n be the number of poles of q. Then $n \geq \max(2g-2, 2)$.*

[2] By suborbifold, we mean as usually done in this theory "locally finite union of suborbifolds".

Proof. The fact that the number n of poles of q must be at least two follows from [11, Thm 4']. To get the lower bound $2g-2$ in the proposition, we use [5, Thm 2] for the closure of the $\mathrm{SL}(2;\mathbb{R})$-orbit $\mathscr{O}$ of (X,q), which is contained in a stratum $\mathrm{P}\mathscr{Q}\big((-1)^n, m_1,\dots,m_r\big)$: we get

$$\lambda_1+\ldots+\lambda_g=\frac{1}{24}\sum_{j=1}^{r}\frac{m_j\,(m_j+4)}{m_j+2}-\frac{n}{8}+\frac{\pi^2}{3}C_{\mathrm{area}}\Big(\overline{O}\Big),$$

where $C_{\mathrm{area}}\Big(\overline{O}\Big)$ is a Siegel–Veech constant of the locus $\overline{\mathscr{O}}$ which is nonnegative. Thus, if $\lambda_1+\ldots+\lambda_g=0$,

$$\sum_{j=1}^{r}\frac{m_j\,(m_j+4)}{m_j+2}\le 3n.$$

Since $(\sum m_j)-n=4g-4$,

$$2g-2\le\sum_{j=1}^{r}\frac{m_j}{m_j+2}+2g-2\le n. \qquad \square$$

REMARK 3. We will see that this bound is asymptotically sharp at the end of this section.

We now focus on the case of pillow-tiled surfaces. Let us start with a technical result:

PROPOSITION 7. *Let X be a Riemann surface of genus g, $\mathrm{B}(t_0,\varepsilon)$ a small ball in $\mathbb{C}\smallsetminus\{0,1,\infty\}$, and $\varphi\colon \mathrm{B}(t_0,\varepsilon)\times X\to\mathbb{P}^1$ be a holomorphic map satisfying the following conditions:*

1. *For any t in $\mathrm{B}(t_0,\varepsilon)$, φ_t is nonconstant and the branching locus of φ_t is $\{0,1,\infty,t\}$.*
2. *The configuration of the ramification points of φ_t remains constant with t.*

If d is the degree of the branched covers φ_t, then $d\ge 3(g-1)$.

Proof. For any x in X, let $s(x)=\frac{\partial}{\partial t}_{|t=t_0}\varphi_t(x)\in \mathrm{T}_{\varphi_{t_0}(x)}\mathbb{P}^1$. Then s is a holomorphic section of the holomorphic line bundle $\varphi_{t_0}^*\mathrm{T}\mathbb{P}^1$. Let x_0 be a ramification point of φ_{t_0} such that $\varphi_{t_0}(x_0)=0$. Let us assume that $s(x_0)\neq 0$. By the Implicit-Function Theorem, the equation $\varphi_t(x)=0$ has a unique solution $(t(x),x)$ depending holomorphically on x for (t,x) near (t_0,x_0). Since $\varphi_{t(x)}(x)=0$, we get

$$\partial_t\varphi(t(x),x)\,t'(x)+(\varphi_{t(x)})'(x)=0.$$

By hypothesis, x is a ramification point of $\varphi_{t(x)}$, i.e., $(\varphi_{t(x)})'(x)=0$. Besides, since $\partial_t\varphi(t(x),x)\to s(x_0)$ as $x\to x_0$, t' vanishes. Hence $\varphi_{t_0}(x)$ vanishes for x near x_0, so that φ_{t_0} is constant and we get a contradiction. It follows that s vanishes at x_0. The same result also holds over any ramification point of φ_{t_0} lying over 1 and ∞. Lastly, if $\psi_t(x)=\varphi_t(x)-t$, the argument we used proves that for any ramification point x of ψ_{t_0} lying over 0, $\frac{\partial}{\partial t}_{|t=t_0}\psi_t(x)=0$, which means that $s(x)=1$. In particular s is nonzero.

We can now decompose the ramification divisor $\mathscr{R}$ of the branched covering φ_{t_0} (which is the divisor of the section $d\varphi_{t_0}$ of $\varphi_{t_0}^*\mathrm{T}\mathbb{P}^1$) as the sum $\mathscr{R}_0+\mathscr{R}_1+$

$\mathscr{R}_\infty + \mathscr{R}_t$. Besides, we can assume that $\deg\mathscr{R}_t$ is smaller than $\deg\mathscr{R}_0$, $\deg\mathscr{R}_1$ and $\deg\mathscr{R}_\infty$, otherwise we move the points $0, 1, \infty$ and t by a suitable homographic transformation. Thanks to the Riemann–Hurwitz formula, we have

$$\deg\mathscr{R} = 2(g+d-1).$$

Now s is a nonzero section of the line bundle $\mathscr{L} = \varphi_{t_0}^* \mathrm{T}\mathbb{P}^1(-\mathscr{R}_0 - \mathscr{R}_1 - \mathscr{R}_\infty)$; hence

$$0 \le \deg\mathscr{L} = 2d - \deg\mathscr{R} + \deg\mathscr{R}_t \le 2d - \frac{3}{4}\deg\mathscr{R} = \frac{d}{2} - \frac{3g}{2} + \frac{3}{2}\cdot$$

The result follows. □

COROLLARY 3. *Let (X, q, π) be a pillow-tiled surface of genus g whose Teichmüller disk lies in the Forni locus, and let d be the degree of π. If the ramification locus of π consists of 4 points, then $d \ge 3(g-1)$.*

We end this section by proving that the estimate $n \ge 2g-2$ of Proposition 6 is asymptotically sharp and that the condition on the ramification locus of π in Corollary 3 is necessary.

Let p be a prime number, and let Y be a cyclic covering of degree p of the projective line fully ramified at three distinct points z_1, z_2 and z_3 obtained by taking the Riemann surface of the polynomial

$$w^p - (z-z_1)^{a_1}(z-z_2)^{a_2}(z-z_3)^{a_3},$$

where $1 \le a_1, a_2, a_3 \le p-1$ and $a_1 + a_2 + a_3 = p$. Then the genus of Y is $\frac{p-1}{2}\cdot$ If q is the pull-back of a meromorphic differential on the sphere with four simple poles at z_1, z_2, z_3 and another point z_4, then $\pi^* q$ has exactly p poles, and three zeros of order $p-2$. Therefore, we have

$$\underbrace{n}_{p} = \underbrace{2g-2}_{p-3} + \underbrace{\sum_j \frac{m_j}{m_j+2}}_{3-\frac{6}{p}} + \underbrace{\frac{\pi^2}{3} C_{\text{area}}}_{\frac{6}{p}}\cdot$$

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JULIEN GRIVAUX <jgrivaux@math.cnrs.fr>: I2M, Université d'Aix-Marseille, 39 rue F. Joliot-Curie, 13453 Marseille Cedex 20, France

PASCAL HUBERT <pascal.hubert@univ-amu.fr>: I2M, Université d'Aix-Marseille, 39 rue F. Joliot-Curie, 13453 Marseille Cedex 20, France