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Déserti, Julie (F-PARIS7-IMJ); **Grivaux, Julien** (F-PARIS6-IMJ)
Automorphisms of rational surfaces with positive entropy. (English summary)
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A smooth complex surface is said to be rational if it is birationally equivalent to the projective plane $\mathbf{CP}^2$. This paper involves the construction of automorphisms of rational surfaces which have positive entropy. Infinite families of rational surfaces carrying automorphisms of positive entropy have been discovered by two different approaches. One approach was given by C. T. McMullen [Publ. Math. Inst. Hautes Études Sci. No. 105 (2007), 49–89; MR2354205 (2008m:37076)]. Subsequent work by J. Diller [Michigan Math. J. **60** (2011), no. 2, 409–440; MR2825269 (2012i:14018)] gave an exhaustive account of the possibilities for automorphisms produced by quadratic maps with invariant curves. More recently T. Uehara ["Rational surface automorphisms with positive entropy", preprint, arXiv:1009.2143] has considered a higher degree case.

The other approach was given by K. Kim and the reviewer [Michigan Math. J. **54** (2006), no. 3, 647–670; MR2280499 (2008k:32054); J. Geom. Anal. **19** (2009), no. 3, 553–583; MR2496566 (2010g:37065)]. The second of these papers also showed that a rational surface automorphism of positive entropy does not need to have an invariant curve, and in fact the existence of an invariant curve seems rather special.

The paper under review presents a method for constructing rational surface automorphisms which is somewhat related to the second approach mentioned above. This approach starts with a well-chosen birational map Φ of $\mathbf{CP}^2$. The objective is to find $\varphi\in \mathrm{PGL}(3,\mathbf{C})$ and an (iterated) blowup $\pi\colon X\to\mathbf{CP}^2$ such that the lift f_X of $f:=\varphi\circ\Phi$ will induce an automorphism of X.

In ["Transformations birationnelles de petit degré", preprint, arXiv:0811.2325], D. Cerveau and J. Déserti gave a classification of the cubic birational maps of the plane, modulo right and left action by linear transformations. Using this classification, the authors select two cubic maps Φ_3 and f which are "promising" in the sense that the exceptional locus is relatively uncomplicated. They apply their approach to show that there are linear maps L and M such that $F_1:=L\circ\Phi_3$ and $F_2:=M\circ f$ both produce rational surface automorphisms of the same positive entropy $\log((3+\sqrt{5})/2)$. This subject has been treated further by J. Blanc ["Dynamical degrees of (pseudo)-automorphisms fixing cubic hypersurfaces", preprint, arXiv:1204.4256].

It has been noted by Kim and the reviewer [Math. Ann. **348** (2010), no. 3, 667–688; MR2677899 (2012a:32014)] that there are nontrivial continuous families of rational surfaces X_α and automorphisms $f_\alpha\in\mathrm{Aut}(X_\alpha)$ with positive entropy. In fact, the parameter α can be taken inside a space with arbitrarily large dimension, and it is shown that the correspondence of dynamical systems $\alpha\to(X_\alpha,f_\alpha)$ is fully nontrivial, in the sense of smooth conjugacy. The concept of nontriviality is that for distinct parameters α_1 and α_2, the dynamical systems $(X_{\alpha_1},f_{\alpha_1})$ and $(X_{\alpha_2},f_{\alpha_2})$ are not conjugate. The precise choices of conjugacy—topological, smooth, or holomorphic—will give different concepts of nontriviality. The authors approach this issue by formalizing the more abstract idea of deformation spaces of manifolds and automorphisms. They work with holomorphic conjugacy and propose a method for determining the number of effective parameters of a family.

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