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Les exposants de Liapounoff du flot de Teichmüller (d'après Eskin-Kontsevich-Zorich). (French. French summary) [The Lyapunov exponents of Teichmüller flow (following Eskin-Kontsevich-Zorich)]
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The purpose of the paper under review is to explain the main ideas and the main ingredients of the involved and delicate work of A. Eskin, M. Kontsevich and A. Zorich concerning the sum of the positive Lyapunov exponents of the so-called Kontsevich-Zorich cocycle acting on the first cohomology spaces of translation surfaces [Publ. Math. Inst. Hautes Études Sci. **120** (2014), 207–333; MR3270590].

Let us fix a genus $g \geq 2$; the space of complex structures $\mathcal{M}_g$ on the closed surface of genus g can be given a complex analytic structure so that the periods define analytic functions. Its universal covering is the genus g Teichmüller space $\mathcal{T}_g$, a contractible Kobayashi hyperbolic complex manifold. The geodesic flow on $\mathcal{T}_g$ descends to $\mathcal{M}_g$ as the Teichmüller flow. The space of translation surfaces corresponds to the fiber-bundle of 1-holomorphic differentials over $\mathcal{M}_g$. Each translation surface X contains finitely many singularities of different multiplicities which are described by the Gauss-Bonnet theorem. The data Σ consisting of the number of singularities together with their multiplicities define the strata of this bundle; they are locally modeled on the relative cohomology space $H^1(X,\Sigma,\mathbf{C})$. The Teichmüller flow preserves each (connected component of each) stratum (of surfaces with fixed area), and its action is ergodic with respect to the (finite) measure ν coming from the Lebesgue measure on the cohomology space.

The Kontsevich-Zorich cocycle can be defined as follows. Let U be a small neighborhood of a translation surface X in its stratum; given a cohomology class $\alpha \in H^1(X,\mathbf{C})$, we push it by parallel transportation until it returns into U: this defines a new class α' in $H^1(X,\mathbf{C})$; this correspondence defines this cocycle. The Kontsevich-Zorich cocycle is particularly important because it governs the deviations of ergodic means of linear flows on flat surfaces. Oseledets' theorem provides us with the existence of Lyapunov exponents for this cocycle. They are known to be all distinct, non-zero, and symmetric with respect to the origin by the works of G. Forni [Ann. of Math. (2) **155** (2002), no. 1, 1–103; MR1888794] and of A. Avila and M. Viana [Acta Math. **198** (2007), no. 1, 1–56; MR2316268].

We now introduce a Siegel-Veech constant C_{SV} which solves a counting problem at the first order analogous to the Gauss circle problem. Given a flat surface X and $R>0$, let C_R denote the collection of homotopy classes $[\gamma]$ of simple closed geodesics of length at most R; each class spans a flat annulus $A[\gamma]\subset X$. Let us consider the counting function

$$N(X,R)=\sum_{[\gamma]\in C_R}\frac{\mathrm{Area}(A[\gamma])}{\mathrm{Area}(X)}\,.$$

This number behaves like R^2 and the Siegel-Veech constant C_{SV} governs its asymptotic behavior in the sense that $N(X,R)\sim \pi C_{SV}R^2$ holds for a generic surface X according to [Y. Vorobets, in *Algebraic and topological dynamics*, 205–258, Contemp. Math., 385, Amer. Math. Soc., Providence, RI, 2005; MR2180238].

The central result in this Exposé is an explicit formula relating the sum of the positive Lyapunov exponents to the Siegel-Veech constant. One consequence of this formula is that the sum of these positive Lyapunov exponents is a rational number: this was previously conjectured by Kontsevich in 1997.

The starting point of the proof is a formula due to Forni, improving on Kontsevich, which relates the sum of the positive Lyapunov exponents to the mean of the curvature of the Hodge bundle in the direction of the so-called Teichmüller disks. While the situation is rather well understood on the Teichmüller curves, this is far from being the case on the whole stratum: a careful analysis of the degenerations of the Hodge structure has to be made close to the singularities. This is done through an ad hoc analytic Riemann-Roch theorem which provides an interpretation of the curvature of the Hodge bundle in terms of the zeta-function of the Laplacian. The authors are then led to understand these metric objects. *Peter Haïssinsky*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.