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Infinitesimal deformations of rational surface automorphisms. (English summary)
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The beauty of the fascinating and exceptionally well-written paper under review resides in the pleasant flavor coming from blending algebraic geometry with the topological theory of dynamical systems. This marvelous interaction has become particularly trendy in the last decade, due especially to the rapid development of the algebraic theory of holomorphic foliations.

This specific article deals with the relevant goal of producing explicit examples of automorphisms of complex rational surfaces having a positive topological entropy. The latter is a notion which is purely topological in nature.

To give a feeling to the non-specialist, any (non-trivial) automorphism of X gives rise to a dynamical system, where the f-orbit of $x \in X$ is the sequence of iterated $(x, f(x), f^2(x), \dots)$. The topological entropy, roughly speaking, measures the asymptotic average rate of growing of distinguishable orbit segments with respect to smaller and smaller neighborhoods (at the limit, in an arbitrarily small neighborhood): see Section 4.4 of S. Cantat's survey "Dynamics of automorphisms of compact complex surfaces" [in *Frontiers in complex dynamics*, 463–514, Princeton Math. Ser., 51, Princeton Univ. Press, Princeton, NJ, 2014; MR3289919].

As the author puts it, however, it is difficult to construct examples, in spite of the fact that they curiously occur in holomorphic families of arbitrarily large dimension. That leads naturally to the study of the deformation theory of *discrete dynamical systems*, understood as pairs (X, f), where X is a rational surface and f is a biholomorphic map (i.e., holomorphic, invertible with holomorphic inverse). The deformation theory of the rational surface X alone, forgetting f, is governed by the Kuranishi theory, which guarantees the existence of a versal deformation space B_X, which is universal if X does not possess non-zero holomorphic vector fields. This is the case considered in the paper under review. Now, a fixed automorphism f of X naturally acts on B_X. Its restriction to the fixed locus Z_f turns out to be universal for the deformation functor of pairs. The natural thing to do for counting the number of parameters is to look at the Zariski tangent space of Z_f. This can be naturally identified with the first cohomology group of the tangent bundle $H^1(X, TX)$, thus acted on by the differential f_* of f. On the other hand, the pullback f^* acts on divisors on the surfaces and the action descends to one of the Neron-Severi group. The latter, in case X is projective, can be identified with the group of divisors modulo numerical equivalence.

The main result of the paper is Theorem 3.11. A few preliminaries are necessary in order to state it. If X is a rational surface, with $K_X^2 < 0$ endowed with an automorphism f such that $|-K_X| = \{C\}$ for an irreducible curve C, let $Q_f(x)$ be the characteristic polynomial of f_*, as an endomorphism of $H^1(X, TX)$, and a_f the multiplier of f, i.e., the induced action of f on the complex line. Then the theorem computes the expression of $Q_f(x)$ in terms of the characteristic polynomial of f^* and the multiplier of f in the case when C is cuspidal or smooth. The proof of the statement is provided in Section 3, at the end of a sequence of ancillary results, some of which are interesting on their own. The main result is applied to prove Theorem 3.16, concerning quadratic birational maps leaving a cuspidal curve globally invariant, and Theorem 3.17. The latter deals with exhibitions of automorphisms of the projective plane fixing a smooth elliptic curve according to an example of J. Blanc [Indiana Univ. Math. J. **62** (2013), no. 4, 1143–1164; MR3179687]. Theorem 3.16 supplies the first example of rigid rational surface automorphisms with positive entropy and Theorem 4.9 shows the example of a rational Kummer surface with an infinite family of rigid automorphisms, each with positive topological entropy.

The paper ends by highlighting the need to develop machinery to construct explicit bases of $H^1(X, TX)$, to deal with the open and not trivial problem of computing the action of f_* on X, in the situation when X is obtained by $\mathbf{P}^2$, after resolving an explicit Cremona transformation.

A rich reference list concludes the paper, helping the reader to find a suitable path to acquire the necessary prerequisites. *Letterio Gatto*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.