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The Ext algebra of a quantized cycle. **(English, French summaries)**
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This interesting paper gives a Lie theoretic study of the derived Ext algebra associated to a tame quantized cycle in a complex manifold or a smooth algebraic variety of characteristic 0.

Let me explain the background, following the introduction of this paper. The Hochschild-Kostant-Rosenberg (HKR) isomorphism has a version in the level of derived categories, which this paper calls the geometric HKR isomorphism. Let X be a smooth scheme or a complex manifold. Then there is an isomorphism of the sheaf of polyvector fields on X and the derived Hom sheaf of the diagonal of X. It is actually an isomorphism of algebras with respect to the wedge products and the Yoneda products. This algebra isomorphism is given by the composition of the global cohomological HKR isomorphism with the twist by the square root $\sqrt{\mathrm{Td}(X)}$ of the Todd class.

One can reformulate this fact in terms of Lie theoretic language (see Theorem 5.9 for details): the shifted tangent sheaf $T_X[-1]$ is a Lie algebra object in the derived category $D^b(X)$, the universal enveloping algebra $U(T_X[-1])$ is the derived Hom sheaf, and the geometric HKR isomorphism coincides with the PBW isomorphism. Moreover the algebra isomorphism, which was a twist of the HKR isomorphism by $\sqrt{\mathrm{Td}(X)}$, coincides with a Duflo isomorphism for the Lie algebra $T_X[-1]$.

The main body of this paper proposes an extension of the above Lie theoretic description of the geometric HKR isomorphism for a smooth scheme X to the setting of a pair $X \subset Y$ of a smooth scheme Y and a cycle X. In a Lie theoretic viewpoint, one is considering a pair $\mathfrak{h} \subset \mathfrak{g}$ of Lie algebras. Precisely speaking, one needs the condition called "tame quantized" for the cycle X. This condition has a natural Lie theoretical interpretation that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as $\mathfrak{h}$-modules and that $[\pi_{\mathfrak{h}}(\mathfrak{m}, \mathfrak{m}), \mathfrak{m}] = 0$.

The main Theorem A states that the shifted normal sheaf $N_{X/Y}[-1]$ is a Lie algebra object in $D^b(X)$, and the associated universal enveloping algebra has an explicit description.

Theorem B states that the quantized cycle class (X, σ) (see §4.3.1 for details) is the Duflo element of the Lie algebra $N_{X/Y}[-1]$. This result was originally found by S. Yu [Adv. Math. **352** (2019), 297–325, doi:10.1016/j.aim.2019.06.003], although a more Lie theoretic proof is given in this paper.

The text is clearly written and the introduction is very readable. I recommend this article to those interested in the representation theoretic study of derived categories of varieties.

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References

1. D. Arinkin & A. Căldăraru – "When is the self-intersection of a subvariety a fibration?", *Adv. Math.* **231** (2012), no. 2, p. 815–842. MR2955193
2. F. A. Berezin – "Some remarks about the associated envelope of a Lie algebra", *Funct. Anal. Appl.* **1** (1967), no. 2, p. 91–102. MR0219671
3. N. Bourbaki – *Algebra I. Chapters 1–3*, Elements of Mathematics, Springer-Verlag, Berlin, 1998. MR1727844

4. D. Calaque & M. Van den Bergh – "Hochschild cohomology and Atiyah classes", *Adv. Math.* **224** (2010), no. 5, p. 1839–1889. MR2646112
5. D. Calaque, A. Căldăraru & J. Tu – "PBW for an inclusion of Lie algebras", *J. Algebra* **378** (2013), p. 64–79. MR3017014
6. D. Calaque, A. Căldăraru & J. Tu – "On the Lie algebroid of a derived self-intersection", *Adv. Math.* **262** (2014), p. 751–783. MR3228441
7. D. Calaque & C. A. Rossi – *Lectures on Duflo isomorphisms in Lie algebra and complex geometry*, EMS Lect. in Math., European Mathematical Society (EMS), Zürich, 2011. MR2816610
8. P. Deligne & J. W. Morgan – "Notes on supersymmetry (following Joseph Bernstein)", in *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, American Mathematical Society, Providence, RI, 1999, p. 41–97. MR1701597
9. M. Duflo – "Opérateurs différentiels bi-invariants sur un groupe de Lie", *Ann. Sci. École Norm. Sup. (4)* **10** (1977), no. 2, p. 265–288. MR0444841
10. M. Duflo – "Open problems in representation theory of lie groups", in *Conference on Analysis on homogeneous spaces (Katata, Japan)* (T. Oshima, ed.), 1986, p. 1–5.
11. J. Grivaux – "On a conjecture of Kashiwara relating Chern and Euler classes of O-modules", *J. Differential Geom.* **90** (2012), no. 2, p. 267–275. MR2899876
12. J. Grivaux – "The Hochschild-Kostant-Rosenberg isomorphism for quantized analytic cycles", *Internat. Math. Res. Notices* (2014), no. 4, p. 865–913. MR3168398
13. J. Grivaux – "Derived geometry of the first formal neighborhood of a smooth analytic cycle", `arXiv: 1505.04414`, 2015.
14. G. Hochschild, B. Kostant & A. Rosenberg – "Differential forms on regular affine algebras", *Trans. Amer. Math. Soc.* **102** (1962), p. 383–408. MR0142598
15. M. Kapranov – "Rozansky-Witten invariants via Atiyah classes", *Compositio Math.* **115** (1999), no. 1, p. 71–113. MR1671737
16. M. Kashiwara & P. Schapira – *Deformation quantization modules*, Astérisque, vol. 345, Société Mathématique de France, Paris, 2012. MR3012169
17. M. Kontsevich – "Deformation quantization of Poisson manifolds", *Lett. Math. Phys.* **66** (2003), no. 3, p. 157–216. MR2062626
18. J.-L. Loday & M. Ronco – "Combinatorial Hopf algebras", in *Quanta of maths*, Clay Math. Proc., vol. 11, American Mathematical Society, Providence, RI, 2010, p. 347–383. MR2732058
19. J.-L. Loday & B. Vallette – *Algebraic operads*, Grundlehren Math. Wiss., vol. 346, Springer-Verlag, Berlin Heidelberg, 2012. MR2954392
20. N. Markarian – "The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem", *J. London Math. Soc. (2)* **79** (2009), no. 1, p. 129–143. MR2472137
21. A. C. Ramadoss – "The big Chern classes and the Chern character", *Internat. J. Math.* **19** (2008), no. 6, p. 699–746. MR2431634
22. A. C. Ramadoss, "The relative Riemann-Roch theorem from Hochschild homology", *New York J. Math.* **14** (2008), p. 643–717. MR2465798
23. D. Toledo & Y. L. L. Tong – "Duality and intersection theory in complex manifolds. I", *Math. Ann.* **237** (1978), no. 1, p. 41–77. MR0506654
24. S. Yu – "Todd class via homotopy perturbation theory", `arXiv:1510.07936`, 2015.

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.