Calaque, Damien; Grivaux, Julien

The Ext algebra of a quantized cycle. (English. French summary) Zbl 07033365

J. Éc. Polytech., Math. 6, 31-77 (2019).

When given a quantized analytic cycle (X,σ) in Y, *Sh. Yu* ["Todd class via homotopy perturbation theory", Preprint, arXiv:1510.07936] discovered a geometric condition by computing the quantized cycle of a closed embedding of complex manifolds using homotopy perturbation theory. Damien Calaque and Julien Grivaux give a categorical Lie-theoretic interpretation of this condition, which involves the second formal neighborhood of X in Y.

A quantized cycle (X,σ) is tame if $\sigma^*\mathrm{N}_{X/Y}$ extends to a locally free sheaf at the second order. If this condition is satisfied, the authors prove that the derived Ext algebra $\mathcal{RH}om_{\mathcal{O}_Y}(\mathcal{O}_X,\mathcal{O}_X)$ is isomorphic to the universal enveloping algebra $U(\mathrm{N}_{X/Y}[-1])$ of the shifted normal bundle $\mathrm{N}_{X/Y}[-1]$, endowed with a specific Lie structure (Theorem A, page 34): assuming that (X,σ) is a tame quantized cycle in Y, the class α defines a Lie coalgebra structure on the shifted conormal bundle $\mathrm{N}^*_{X/Y}[1]$, which gives a Lie algebra structure on $\mathrm{N}_{X/Y}[-1]$. Moreover, $\mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X,\mathcal{O}_X)$ and $\mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{O}_X,\mathcal{O}_X)$ are algebra objects in the derived category $\mathrm{D}^{\mathrm{b}}(X)$, and the following diagrams

$$\begin{array}{ccc}
\sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X,\mathcal{O}_X) & \longrightarrow & \mathcal{RH}om^{\ell}_{\mathcal{O}_Y}(\mathcal{O}_X,\mathcal{O}_X) \\
 & & \simeq\big\downarrow \mathrm{HKR} \\
\mathrm{HKR}\big\downarrow\simeq & & \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \\
 & & \simeq\big\downarrow \mathrm{PBW} \\
\mathrm{T}(\mathrm{N}_{X/Y}[-1]) & \longrightarrow & \mathrm{U}(\mathrm{N}_{X/Y}[-1])
\end{array}$$

and

$$\begin{array}{ccc}
\sigma_*\mathcal{RH}om_{\mathcal{O}_S}(\mathcal{O}_X,\mathcal{O}_X) & \longrightarrow & \mathcal{RH}om^{r}_{\mathcal{O}_Y}(\mathcal{O}_X,\mathcal{O}_X) \\
 & & \simeq\big\downarrow \text{dual HKR} \\
\text{dual HKR}\big\downarrow\simeq & & \mathrm{S}(\mathrm{N}_{X/Y}[-1]) \\
 & & \simeq\big\downarrow \mathrm{PBW} \\
\mathrm{T}(\mathrm{N}_{X/Y}[-1]) & \longrightarrow & \mathrm{U}(\mathrm{N}_{X/Y}[-1])
\end{array}$$

commute, where all horizontal arrows are algebra morphisms.

The authors also give a new Lie-theoretic proof of S. Yu's result for the tame quantized cycle class (Theorem B, page 35): letting (X,σ) to be a tame quantized cycle in Y, the quantized cycle class of (X,σ) defined by *J. Grivaux* [Int. Math. Res. Not. 2014, No. 4, 865–913 (2014; Zbl 1312.14027)] is the Duflo element of the Lie algebra object $\mathrm{N}_{X/Y}[-1]$.

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MSC:

14F05 Sheaves, derived categories of sheaves, etc.
14B20 Formal neighborhoods
17B35 Universal enveloping Lie (super)algebras
16S30 Universal enveloping algebras of Lie algebras (associative)
14C99 Cycles and subschemes

Keywords:

closed embeddings; formal neighborhoods; Todd class; Ext algebra; derived categories; Lie algebras; en-

veloping algebras; Duflo element; PBW isomorphism

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