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**Derived geometry of the first formal neighbourhood of a smooth analytic cycle. (English summary)**
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The paper under review starts from the Hochschild-Kostant-Rosenberg (HKR) isomorphism, which states that for a complex manifold or smooth scheme $X$ the diagonal injection $\delta$ induces an isomorphism

$$\mathbb{L}\delta^*(\delta_*\mathcal{O}_X) \simeq \bigoplus_{p=0}^{\dim X} \Omega_X^p[p].$$

This construction was extended to various other closed immersions starting with the work of Kashiwara and Căldăraru and Arinkin. In this work the author considers first-order thickenings.

Let $k$ be a field of characteristic 0 and let $(X, \mathcal{O}_X)$ be a $k$-ringed space. Let $k_X$ be the sheaf of locally constant $k$-valued functions on $X$ and let $\mathcal{I}$ be a locally-free sheaf of finite rank on $X$. An infinitesimal thickening of $X$ by $\mathcal{I}$ is a sheaf $\mathcal{O}_S$ of $k_X$-algebras such that there is the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_S \to \mathcal{O}_X \to 0,$$

where $\mathcal{I}^2 = 0$.

This condition implies that $\mathcal{O}_S$ is locally isomorphic with the trivial $k_X$-extension of $\mathcal{O}_X$ by $\mathcal{I}$. Geometrically if one considers $S = (X, \mathcal{O}_S)$ as a ringed space then there is an immersion $j\colon X \to S$ that admits locally a right inverse.

Let $X$ now be a smooth $k$-scheme and let $C^-(X)$ be the category of bounded sheaf complexes. For any complex $\mathcal{K}_* \in C^-(X)$, the author introduces a morphism $\Theta_{\mathcal{K}_*}\colon j * \mathcal{K}_* = \mathrm{Tor}^0_{\mathcal{O}_S}(\mathcal{K}_*, \mathcal{O}_S) \to \mathrm{Tor}^1_{\mathcal{O}_S}(\mathcal{K}_*, \mathcal{O}_S)[2]$ in $\mathcal{D}^-(X)$. The complex $\mathcal{K}_*$ is assumed to be a bounded complex with $\mathrm{Tor}^1_{\mathcal{O}_S}(\mathcal{K}_*, \mathcal{O}_X)$ quasi-isomorphic with zero.

The author proves that the following are equivalent:

(1) $\Theta_{\mathcal{K}_*}$ vanishes;
(2) the morphism $\mathbb{L}j^*\mathcal{K}_* \to j^*\mathcal{K}_*$ admits a right inverse in $\mathcal{D}^-(X)$;
(3) there is a bounded admissible complex $\mathcal{L}_*$ and a morphism $\mathcal{L}_* \to \mathcal{K}_*$ in $\mathcal{D}^b(X)$ such that the composition $\mathbb{L}j^*\mathcal{L}_* \to \mathbb{L}j^*\mathcal{K}_* \to j^*\mathcal{K}_*$ is an isomorphism in $\mathcal{D}^-(X)$.

A criterion for the vanishing of $\Theta_{\mathcal{K}_*}$ is also given. Finally, the author constructs bounded approximations of the functor $\mathbb{L}j^*j_*$ by an exact endofunctor $H(\mathcal{V}) =: \mathrm{cone}(\Omega^1_X \otimes \mathcal{V}_* \to P^1_X(\mathcal{V}_*))$ which is a Fourier-Mukai transform. $H$ comes with a natural morphism to the identity functor. Let $H^{[n]}$ be the equalizer of the $n$ natural maps from $H^n \to H^{n-1}$ induced by this morphism.

Then the functor $\mathbb{L}j^*j_*$ is isomorphic with $\varprojlim H^{[n]}$. When $S$ is globally trivial, the generalized HKR isomorphism constructed by Căldărau and Arinkin is recovered.

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## References

1. P. Bressler, R. Nest, B. Tsygan, Riemann-Roch theorems via deformation quanti-

zation. I, II, Adv. Math. 167 (1) (2002) 1–25, 26–73. MR1901245
2. R.-O. Buchweitz, H. Flenner, The global decomposition theorem for Hochschild (co-)homology of singular spaces via the Atiyah-Chern character, Adv. Math. 217 (1) (2008) 243–281. MR2357327
3. D. Calaque, C.A. Rossi, Lectures on Duflo Isomorphisms in Lie Algebra and Complex Geometry, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2011. MR2816610
4. D. Calaque, M. Van den Bergh, Hochschild cohomology and Atiyah classes, Adv. Math. 224 (5) (2010) 1839–1889. MR2646112
5. D. Calaque, C.A. Rossi, M. Van den Bergh, Căldăraru's conjecture and Tsygan's formality, Ann. of Math. (2) 176 (2) (2012) 865–923. MR2950766
6. D. Calaque, A. Căldăraru, J. Tu, On the Lie algebroid of a derived self-intersection, Adv. Math. 262 (2014) 751–783. MR3228441
7. A. Căldăraru, The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism, Adv. Math. 194 (1) (2005) 34–66. MR2141853
8. A. Dima, A. Căldăraru, When is the self-intersection of a subvariety a fibration? Adv. Math. 231 (2) (2012) 815–842. MR2955193
9. Dima Arinkin, Andrei Căldăraru, Marton Hablicsek, On the formality of derived intersections, preprint, 2014.
10. V. Dolgushev, D. Tamarkin, B. Tsygan, The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal, J. Noncommut. Geom. 1 (1) (2007) 1–25. MR2294189
11. J. Grivaux, On a conjecture of Kashiwara relating Chern and Euler classes of o-modules, J. Differential Geom. 90 (2) (2012) 267–275. MR2899876
12. J. Grivaux, Formality of derived intersections, Doc. Math. 19 (2014) 1003–1016. MR3262078
13. J. Grivaux, The Hochschild-Kostant-Rosenberg isomorphism for quantized analytic cycles, Int. Math. Res. Not. IMRN (4) (2014) 865–913. MR3168398
14. G. Hochschild, B. Kostant, A. Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. 102 (1962) 383–408. MR0142598
15. D. Huybrechts, R.P. Thomas, Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes, Math. Ann. 346 (3) (2010) 545–569. MR2578562
16. M. Kapranov, Rozansky-Witten invariants via Atiyah classes, Compos. Math. 115 (1) (1999) 71–113. MR1671737
17. M. Kashiwara, Unpublished letter to Pierre Schapira, 1991.
18. M. Kashiwara, P. Schapira, Categories and Sheaves, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 332, Springer-Verlag, Berlin, 2006. MR2182076
19. M. Kashiwara, P. Schapira, Deformation quantization modules. Astérisque, Société Mathématique de France, 2012. MR3012169
20. M. Kontsevich, Homological algebra of mirror symmetry, in: Proceedings of the International Congress of Mathematicians, Vols. 1, 2, Zürich, 1994, Birkhäuser, Basel, 1995, pp. 120–139. MR1403918
21. M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (3) (2003) 157–216. MR2062626
22. M. Lieblich, Moduli of complexes on a proper morphism, J. Algebraic Geom. 15 (1) (2006) 175–206. MR2177199
23. W. Lowen, Obstruction theory for objects in Abelian and derived categories, Comm. Algebra 33 (9) (2005) 3195–3223. MR2175388
24. N. Markarian, The Atiyah class, Hochschild cohomology and the Riemann-Roch

theorem, J. Lond. Math. Soc. (2) 79 (1) (2009) 129–143. MR2472137
25. D. Quillen, On the (co-) homology of commutative rings, in: Applications of Categorical Algebra, Proc. Sympos. Pure Math., Vol. XVII, New York, 1968, Amer. Math. Soc., Providence, R.I., 1970, pp. 65–87. MR0257068
26. A.C. Ramadoss, The relative Riemann-Roch theorem from Hochschild homology, New York J. Math. 14 (2008) 643–717. MR2465798
27. R.G. Swan, Hochschild cohomology of quasiprojective schemes, J. Pure Appl. Algebra 110 (1) (1996) 57–80. MR1390671
28. B. Toën, The homotopy theory of *dg*-categories and derived Morita theory, Invent. Math. 167 (3) (2007) 615–667. MR2276263
29. B. Toën, Derived algebraic geometry, EMS Surv. Math. Sci. 1 (2) (2014) 153–240. MR3285853
30. C.A. Weibel, An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324
31. A. Yekutieli, The continuous Hochschild cochain complex of a scheme, Canad. J. Math. 54 (6) (2002) 1319–1337. MR1940241
32. S. Yu, The Dolbeault dga of a Formal neighborhood, Thesis (Ph.D.)–The Pennsylvania State University, ProQuest LLC, Ann Arbor, MI, 2013. MR3218038

*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*